(DOD) 11.1.2 CMB LIKELIHOOD

Our first real world example was making a temperature anisotropy map from the time ordered data of an experiment. We will now discuss the anisotropy map and further compressions to less variables containing the same information. We will largely follow the book by Dodelson which also covers galaxy surveys – we will do this, too.

Inflation (and actually our observations) tell us that the temperature anisotropy signal on the sky is gaussian distributed:

$$\text{prob}(\vec{s}\,|\,C_s) = \frac{1}{(2\pi)^{N_p/2}\sqrt{\det C_s}} \exp\left[-\frac{1}{2}\vec{s}^{\,T} C_s^{-1}\vec{s}\right]$$

where $\vec{s}$ is the signal and C_s the signal covariance matrix and N_p the # of pixels. If you only have one single pixel you look at, you would for instance have

$$\text{prob}(s\,|\,C_s) = \frac{1}{\sqrt{2\pi C_s}} \exp\left[-\frac{s^2}{2C_s}\right]$$

Unfortunately, we cannot measure $\vec{s}$, but only estimate the signal Δ, i.e. we measure Δ. If the noise is also Gaussian, we will have

$$\text{prob}(\Delta\,|\,C_s) = \int ds \frac{e^{-\frac{1}{2}\vec{s}^{\,T} C_s^{-1}\vec{s}}}{(2\pi)^{N_p/2}\sqrt{\det C_s}} \frac{e^{-\frac{1}{2}[\vec{\Delta}-\vec{s}]^T C_N^{-1}[\vec{\Delta}-\vec{s}]}}{(2\pi)^{N_p/2}\sqrt{\det C_N}}$$

Let me regroup the exponentials:

$$\exp\left[-\frac{1}{2}\vec{s}^{T} C_s^{-1}\vec{s} - \frac{1}{2}\vec{\Delta}^{T} C_N^{-1}\vec{\Delta} + \frac{1}{2}\vec{\Delta}^{T} C_N^{-1}\vec{s} + \frac{1}{2}\vec{s}^{T} C_N^{-1}\vec{\Delta} - \frac{1}{2}\vec{s}^{T} C_N^{-1}\vec{s}\right.$$

$$= \exp\left[-\frac{1}{2}\vec{s}^{T}\left(C_s^{-1}+C_N^{-1}\right)\vec{s} + \vec{s}^{T} C_N^{-1}\vec{\Delta} - \frac{1}{2}\vec{\Delta}^{T} C_N^{-1}\vec{\Delta}\right]$$

where $\vec{s}^{T} C_N^{-1}\vec{\Delta} = \vec{\Delta}^{T} C_N^{-1}\vec{s}$, because C_N is symmetric.

We have already seen how to perform a Gaussian integral with source $\vec{j}$. Let us use this:

$$\mathrm{prob}(\Delta|C_s) = \frac{(2\pi)^{-N_p}}{\sqrt{\det C_s \cdot \det C_N}} \int ds \exp\left[-\frac{1}{2}\vec{s}^{T}\left(C_s^{-1}+C_N^{-1}\right)\vec{s} + \vec{s}^{T} C_N^{-1}\vec{\Delta}\right] \times \exp\left[-\frac{1}{2}\vec{\Delta}^{T} C_N^{-1}\vec{\Delta}\right]$$

Call $A \equiv C_s^{-1} + C_N^{-1}$; $\vec{j} = C_N^{-1}\vec{\Delta}$, then

$$\mathrm{prob}(\Delta|C_s) = \frac{(2\pi)^{-N_p}}{\sqrt{\det C_s \det C_N}} \int ds \exp\left[-\frac{1}{2}\vec{s}^{T} A\vec{s} + \vec{s}^{T}\vec{j}\right] \exp\left[-\frac{1}{2}\vec{\Delta}^{T} C_N^{-1}\vec{\Delta}\right]$$

$$= \underbrace{\frac{(2\pi)^{-N_p}}{\sqrt{\det C_s \det C_N}} \cdot \frac{(2\pi)^{N_p/2}}{\sqrt{\det A}}}_{V} \exp\left[\frac{1}{2}\vec{j}^{T} A^{-1}\vec{j}\right] \exp\left[-\frac{1}{2}\vec{\Delta}^{T} C_N^{-1}\vec{\Delta}\right]$$

$$= V \cdot \exp\left[\frac{1}{2}\underbrace{\left\{\vec{\Delta}^{T} C_N^{-1\,T}\left[C_s^{-1}+C_N^{-1}\right]^{-1} C_N^{-1}\vec{\Delta} - \vec{\Delta}^{T} C_N^{-1}\vec{\Delta}\right\}}_{*}\right]$$

$$* = \vec{\Delta}^T C_S^{-1} [\mathbb{1} + C_N C_S^{-1}]^{-1} \underline{C_N C_N^{-1}} \vec{\Delta} - \vec{\Delta}^T C_N^{-1} \vec{\Delta}$$

$$= \vec{\Delta}^T C_N^{-1} [\mathbb{1} + C_N C_S^{-1}]^{-1} \vec{\Delta} - \vec{\Delta}^T C_N^{-1} \vec{\Delta}$$

$$= \vec{\Delta}^T \left[C_N^{-1} [\cancel{\mathbb{1}} - C_N C_S^{-1} + C_N C_S^{-1} C_N C_S^{-1} - \ldots] - \cancel{C_N^{-1}} \right] \vec{\Delta}$$

$$= \vec{\Delta}^T \left[- C_S^{-1} + C_S^{-1} C_N C_S^{-1} - C_S^{-1} C_N C_S^{-1} C_N C_S^{-1} + \ldots \right] \vec{\Delta}$$

$$= - \vec{\Delta}^T \left(\mathbb{1} - C_S^{-1} C_N + C_S^{-1} C_N C_S^{-1} C_N - \ldots \right) C_S^{-1} \vec{\Delta}$$

$$= - \vec{\Delta}^T \left[\mathbb{1} + C_S^{-1} C_N \right]^{-1} C_S^{-1} \vec{\Delta}$$

$$= - \vec{\Delta}^T \left[C_S + C_N \right]^{-1} \vec{\Delta}$$

$$\equiv - \Delta^T C^{-1} \vec{\Delta}$$

in addition: $\det C_S \cdot \det C_N \cdot \det [C_S^{-1} + C_N^{-1}]$

$$= \det \{ C_S [C_S^{-1} + C_N^{-1}] \cdot C_N \}$$

$$= \det \{ C_S [C_S^{-1} C_N + \mathbb{1}] \}$$

$$= \det \{ C_N + C_S \} = \det C$$

$$\Rightarrow \text{prob}(\vec{\Delta} | C) = \frac{(2\pi)^{-Np/2}}{\sqrt{\det C}} \exp\left[-\frac{1}{2} \vec{\Delta}^T C^{-1} \vec{\Delta} \right]$$

Which is our final result for the likelihood of Δ given some CMB covariance matrix C with $C \equiv C_S + C_N$

(DD) 11.1.3 Galaxy Surveys

At first sight, galaxy surveys and CMB have not much in common: CMB signal is gaussian, whereas distribution of Galaxies is non-gaussian due to non-linear evolution. In addition, surveys list of discrete objects, CMB a continuous quantity. However, they have much in common, statistically. Define the data pixel i of a survey as

$$\Delta_i \equiv \int d^3x \ \psi_i(\vec{x}) \left[\frac{n(\vec{x}) - \bar{n}(\vec{x})}{\bar{n}(\vec{x})} \right]$$

$n(\vec{x})$: galaxy density

$\langle \vec{n}(\vec{x}) \rangle \equiv \bar{n}(\vec{x})$: expected number for uniform distribution in the survey

Pixilisation scheme determined by ψ:

$$\psi_i^{CIC} = \begin{cases} \bar{n}(\vec{x}) & \text{if } \vec{x} \text{ in } i\text{th subvolume} \\ 0 & \end{cases}$$

"count in cells"

Fourier pixels:

$$\psi_i^{Fourier}(\vec{x}) = \frac{e^{i\vec{k}_i \vec{x}}}{V} \begin{cases} 1 & \text{: inside survey} \\ 0 & \text{: outside survey} \end{cases}$$

Really difficult to come up with likelihood function for Δ: galaxy formation to complicated

However, still valid: $\langle \Delta_i \rangle = 0$

$$\langle \vec{\Delta} \ \vec{\Delta}^T \rangle = C_S + C_N$$

Even without signal, we will have Poisson noise in our survey. Remember that

$$\text{prob}(n) = \frac{\bar{n}^{n} e^{-\bar{n}}}{n!}$$

is the poisson distribution and that

$$\langle n \rangle = \bar{n} \quad ; \langle n^2 \rangle = \bar{n}^2 + \bar{n} \quad ; \langle n^2 \rangle - \langle n \rangle^2 = \bar{n}$$

In fact, it can be shown (see Feldman, Peacock, Kaiser 1994) that for our $\vec{x}$-dependent case,

$$\langle n(\vec{x}) \rangle = \bar{n}(\vec{x})$$

$$\langle n(\vec{x})\, n(\vec{x}') \rangle = \bar{n}(\vec{x})\bar{n}(\vec{x}') + \delta(\vec{x}-\vec{x}')\, \bar{n}(\vec{x})$$

much like the usual Poisson noise above.

So the noise covariance is

$$\langle \Delta_i \Delta_j \rangle_{\text{noise}} \equiv (C_N)_{ij}$$

$$= \int d^3\vec{x}\, d^3\vec{x}'\, \psi_i(\vec{x})\psi_j(\vec{x}') \left\langle \frac{n(\vec{x})n(\vec{x}') - \bar{n}(\vec{x})n(\vec{x}') - \bar{n}(\vec{x}')n(\vec{x}) + \bar{n}(\vec{x})\bar{n}(\vec{x}')}{\bar{n}(\vec{x})\bar{n}(\vec{x}')} \right\rangle$$

$$= \int d^3\vec{x}\, d^3\vec{x}'\, \frac{\psi_i(\vec{x})\psi_j(\vec{x}')}{\bar{n}(\vec{x})\bar{n}(\vec{x}')} \Big[\langle n(\vec{x})n(\vec{x}') \rangle - \bar{n}(\vec{x})\bar{n}(\vec{x}') - \cancel{\bar{n}(\vec{x}')\bar{n}(\vec{x})} + \cancel{\bar{n}(\vec{x})\bar{n}(\vec{x}')} \Big]$$

$$= \int d^3\vec{x}\, d^3\vec{x}'\, \frac{\psi_i(\vec{x})\psi_j(\vec{x}')}{\bar{n}(\vec{x})\bar{n}(\vec{x}')} \Big[\cancel{\bar{n}(\vec{x})\bar{n}(\vec{x}')} + \delta(\vec{x}-\vec{x}')\, \bar{n}(\vec{x}) - \cancel{\bar{n}(\vec{x})\bar{n}(\vec{x}')}$$

$$= \int d^3\vec{x}\, \frac{\psi_i(\vec{x})\psi_j(\vec{x}')}{\bar{n}(\vec{x})} \quad \blacksquare$$