Condensed Matter Theory — problem set 5

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Problem 12: Berry phase

We consider the adiabatic movement of a particle according to the Schrödinger equation $i\hbar\partial_t |\psi\rangle = H(t)|\psi\rangle$ with a time-dependent Hamiltonian $H(t)$. At each instant in time, consider the non-degenerate eigenvalue problem $H(t)|n(t)\rangle = E_n(t)|n(t)\rangle$.

(a) Make the ansatz $|\psi(t)\rangle = \sum_n c_n(t)e^{-i\theta_n(t)}|n(t)\rangle$ with $\theta_n(t) = \frac{1}{\hbar}\int_0^t dt' E_n(t')$ and derive

$$\partial_t c_m(t) = -c_m \langle m|\partial_t m\rangle - \sum_{n\neq m} c_n \frac{\langle m|\partial_t H|n\rangle}{E_n(t) - E_m(t)} e^{i(\theta_m - \theta_n)}. \tag{1}$$

(b) Assume that the Hamiltonian changes slowly in time. In this limit, show that the coefficients of the ansatz are given by $c_m(t) = c_m(0)e^{i\gamma_m(t)}$ with the *Berry phase*

$$\gamma_m(t) = i\int_0^t dt' \langle m(t')|\partial_{t'} m(t')\rangle \ . \tag{2}$$

Prove that the Berry phase $\gamma_m(t)$ is real.

(c) Consider now the Berry phase on a closed curve C parametrized by $\mathbf{r}(t)$ and show that it can be written as

$$\gamma_m = \int_C d\mathbf{r}\cdot\mathbf{A} = \int_S d\mathbf{S}\cdot(\nabla\times\mathbf{A}) \tag{3}$$

where the curl is given by

$$\nabla\times\mathbf{A} = i\sum_{m\neq n}\frac{\langle n|\nabla H|m\rangle\langle m|\nabla H|n\rangle}{(E_n - E_m)^2} \ . \tag{4}$$

(d) Consider a particle in one dimension subject to a periodic electrostatic potential $V(x)$ and a slowly varying time-dependent vector potential $A(t)$,

$$i\hbar\partial_t\psi(t) = \left[\frac{1}{2m}\left(p - \frac{e}{c}A(t)\right)^2 + V(x)\right]\psi(t). \tag{5}$$

Express the Berry phase in terms of Bloch eigenfunctions.

Problem 13: Matsubara sum

Compute the sum

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \tag{6}$$

using the Matsubara technique for fermionic Matsubara frequencies $\omega_n = (2n+1)\pi/\beta$, $n \in \mathbb{Z}$. Temperature is just a formal parameter that will drop out of the final result.

Problem 14: Interaction energy

Consider a translation invariant system of spinless fermions with the Hamiltonian

$$\begin{aligned}\hat{\mathcal{H}} &= \hat{T} + \hat{V} - \mu\hat{N} \\ &= \int d^3x\, \psi^\dagger(x)\Big(\frac{-\nabla^2}{2m} - \mu\Big)\psi(x) + \frac{1}{2}\int d^3x\, d^3x'\, \psi^\dagger(x)\psi^\dagger(x')v(x-x')\psi(x')\psi(x)\,.\end{aligned}$$

Show that

$$\langle \hat{V} \rangle = \sum_k \int dE\, \frac{E + \mu - k^2/2m}{2} A(k,E) f(E) \tag{7}$$

where $f(E) = (e^{\beta E}+1)^{-1}$ is the Fermi function. It is remarkable that the expection value of the two-particle operator $\hat{V}$ for the interaction energy can be expressed in terms of the single-particle spectral function. One can, e.g., do the following steps:

(a) Start with the equation of motion for the Heisenberg operator, $i\partial_t\psi(x,t)$, and explicitly compute the commutator $[\psi(x,t), \hat{V}]$.

(b) Multiply from the left with $\psi^\dagger(x',t')$ and take the expectation value in the limit $t' \to t+0$, $x' \to x$. Then integrate over x and identify $\langle \hat{V} \rangle$.

(c) Express the remaining terms by the Green function $G(xt, x't')$ in the same limit.

(d) Write the Green function using its Fourier components $G(k,\omega)$ and express these in terms of $A(k,E)$ as in the lecture. What happens for a noninteracting system?