Medium-induced Interaction Between Impurities in a Bose-Einstein Condensate

Moritz Drescher, Manfred Salmhofer, and Tilman Enss

Institut für Theoretische Physik, Universität Heidelberg, D-69120 Heidelberg, Germany

We consider two heavy particles immersed in a Bose-Einstein condensate in three dimensions and compute their mutual interaction induced by excitations of the medium. For an ideal Bose gas, the induced interaction is Newtonian up to a shift in distance which depends on the coupling strength between impurities and Bosons. For a real BEC, we find that on short distances, the induced potential is dominated by three-body physics of a single Boson bound to the impurities, leading to an Efimov potential. At large distances of the order of the healing length, a Yukawa potential emerges instead. In particular, we find that both regimes are realized for all impurity-boson couplings and determine the corresponding crossover scales. The transition from the real to the ideal condensate at low gas parameters is investigated.

I. INTRODUCTION

Interactions between particles in fundamental physics emerge by local excitations of fields. When the particles are sufficiently slow, the field's degrees of freedom can be integrated out, leading to an effective action-at-a-distance, which may be described by a distance-dependant potential. The properties of the induced potential depend on properties of the field and become highly non-trivial when the field itself consists of interacting particles.

Recent progress in the creation and control of ultracold gas systems has enabled experiments where the influence of such interacting media of particles on impurities can be studied. In the case where the medium is a Bose-Einstein condensate (BEC), the impurities are known as Bose polarons and have been realized experimentally [1–4] and studied in and out of equilibrium [5–15].

A number of theoretical works have studied interactions induced by a BEC. [16] applies Bogoliubov theory to study how the Efimov hierarchy generated by a single Boson is modified if a whole condensate is present. In [17], a variational wave function with a single Bogoliubov excitation led to a Yukawa potential for weak impurity-boson (IB) coupling and to an Efimov potential near a resonance. A perturbative treatment in momentum space was given in [18]. [19] discusses the case of an ideal condensate, including finite temperature. In [20], general superfluids with a linear dispersion were considered, which, when applied to a BEC, treats the case of impurity separations large compared to the healing length. It was shown that first-order effects in the gas parameter, which would lead to a Yukawa potential, are dominated at these scales by second-order effects and a van-der-Waals potential emerges. [21] consider a soft IB potential while in [22], the impurities are ionic and interact via an r^{-4} potential and in [23], a spinor BEC is investigated. Also related are works on 1d condensates [24–28] and on Fermi polarons [29–33].

In this paper, we consider the experimentally relevant case of a short (though finite) range potential. We employ Gross-Pitaevskii theory (GPT) as well as a recently developed non-local extension (NLGPT, [9]) to be able to treat large deformations of the condensate. We find that
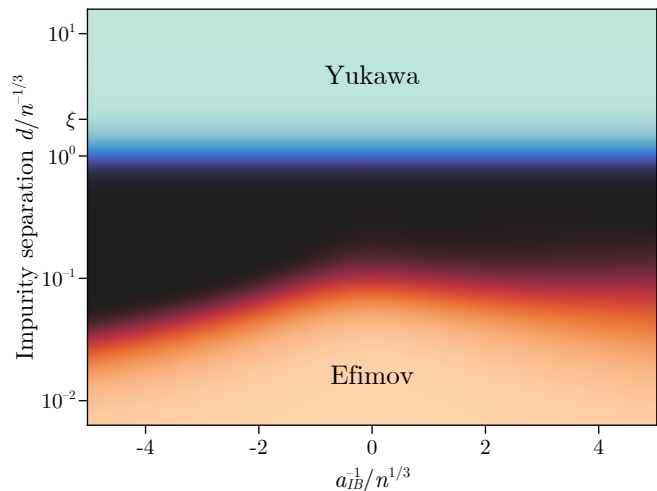

FIG. 1. Sketch of the regimes of Efimov and Yukawa scaling of the induced potential at a gas parameter of $na_{\mathrm{BB}}^3 = 10^{-6}$. At small impurity separations d, a three-body bound state is formed, which has an Efimov scaling $E_{\mathrm{b}} \sim d^{-2}$ for $d \ll |a_{IB}|$ where a_{IB} is the impurity-boson scattering length. At separations comparable to the healing length, the induced potential takes a Yukawa form $\sim \exp(-d\sqrt{2}/\xi)/d$ with the healing length $\xi = 1/\sqrt{8\pi na_{\mathrm{BB}}}$. At even larger separations, a van-der-Waals scaling has been shown to emerge [20] (not shown in figure).

the Efimov and Yukawa scaling of the induced potential, previously found at different coupling strengths [17], in fact exist at every coupling but at different impurity separations as depicted in Fig. 1. For an ideal BEC, on the other hand, the potential is shown to be Newtonian up to a shift in distance. Approaching the ideal gas case by a weakly interacting condensate, we show that the Newtonian potential emerges only at very weak gas parameters of order 10^{-15} in an intermediate distance regime.

II. MODEL

The Hamiltonian for two stationary impurities in a BEC is given by

$$H = \int_x a_x^\dagger \left(-\frac{\Delta}{2m_B} + V_{x-x_{I,1}}^{IB} + V_{x-x_{I,2}}^{IB} \right) a_x \\ + \int_{x,y} a_x^\dagger a_y^\dagger V_{x-y}^{BB} a_x a_y \quad (1)$$

where $a_x^{(\dagger)}$ are the boson annihilation and creation operators at position $x \in \mathbb{R}^3$, m_B the boson mass, V^{IB} and V^{BB} the impurity-boson and boson-boson interaction potentials of scattering lengths a_{IB} and a_{BB}, respectively, and $x_{I,i}$ the impurity positions. Their distance will be denoted $d = |x_{I,2} - x_{I,1}|$. We are looking for the ground state energy E_d of this Hamiltonian at different impurity separations $d = |x_{I,1} - x_{I,2}|$ to compare it with the energy of two individual impurities E_∞ and thus obtain the medium-induced interaction

$$V_{ind}(d) = E_d - E_\infty. \quad (2)$$

A. Basic properties

Before turning to the methods applied to the Hamiltonian, let us recall some elementary analytical results, which serve as a reference.

a. Single-Boson Problem. The three-body problem of a single Boson and two stationary impurities can be solved in the case of an IB contact potential. Clearly, the effective potential formed by the two impurities is the stronger the closer the impurities are. One finds, that a bound state exists whenever $a_{IB}^{-1} > -d^{-1}$ (compared to $a_{IB}^{-1} > 0$ for a single impurity) with binding energy of $E_b = \kappa^2/2m_B$ where

$$\kappa = \frac{1}{a_{IB}} + \frac{1}{d} W_0(e^{-d/a_{IB}}) \quad (3)$$

and W_0 is the principal branch of the Lambert W function, i.e. the inverse of we^w defined on $[-1/e, \infty)$ [29, 32]. At short distances or resonant coupling $a_{IB}^{-1} = 0$, the binding energy takes the form $E_b \sim d^{-2}$, which is reminiscent of Efimov physics and admits a tower of bound states. We may expect to find the same behaviour at small distances in the many-body problem [34], as a single boson gets bound to the impurities and repels the other bosons, c.f. [13].

b. Ideal BEC. In absence of BB interactions, the ground state of (1) is a product state with all bosons being in the ground state of the single-boson problem. If this problem allows for a bound state, the many-body ground state energy will be $-\infty$ as all bosons enter the bound state and the BEC collapses. Otherwise, all bosons are in the zero-mode, i.e. the lowest scattering state, and the ground state energy will be finite [12]. It is not zero, however, because in finite volume Ω, the lowest scattering state has an energy of order $1/\Omega$, which, when occupied N times, does not vanish in the thermodynamic limit.

Again, the problem can be solved explicitly for contact interactions. Regarding the two impurities as forming a single effective potential of scattering length a_{eff}, the ground state energy is given by $E = 4\pi a_{eff} n/2m_B$ (c.f. [12]). The zero mode ϕ^0 defines a_{eff} via $\phi^0(x) \to 1 - a_{eff}/|x|$ as $|x| \to \infty$. It reads[1]

$$\phi_x^0 = 1 - \frac{\alpha}{|x - x_{I,1}|} - \frac{\alpha}{|x - x_{I,2}|}$$

where $\alpha = 1/(a_{IB}^{-1} + d^{-1})$, so we obtain $a_{eff} = 2\alpha$ and

$$E_d = \frac{8\pi n/2m_B}{a_{IB}^{-1} + d^{-1}}.$$

This formula has already been derived by a different method in [19] and leads to a "shifted Newtonian" induced attractive potential

$$V_{ind}(d) = E_d - E_\infty = -\frac{8\pi a_{IB}^2 n}{2m_B} \frac{1}{a_{IB} + d}$$

whenever $a_{eff} < 0$, i.e. $a_{IB} < 0$ and $|a_{IB}| < d$ (otherwise, $V_{ind} = -\infty$ because a bound state exists which can be entered by all bosons).

B. Methods

For the full interacting problem, we employ Gross-Pitaevskii theory (GPT) as well as its non-local extension (NLGPT) developed recently in [9].

GPT. If the boson density varies on a length scale larger than the BB interaction range, a local density approximation can be applied, and the zero temperature condensate can be described by a single condensate wave function ϕ. Using that the ground state energy density of a BEC is $4\pi a_{BB} n^2/2m_B$, this leads to the well-known Gross-Pitaevskii energy functional

$$E_{GP} = \int_x \overline{\phi_x} \left(-\frac{\Delta}{2m_B} + V_{x-x_{I,1}}^{IB} + V_{x-x_{I,2}}^{IB} \right) \phi_x \\ + \frac{4\pi a_{BB}}{2m_B} \int_x |\phi_x|^4. \quad (4)$$

Most results of this work are obtained by minimizing this functional. To justify the local density approximation, the IB potentials must not be too sharp [9]: For instance for a contact potential, ϕ would have poles $1/|x - x_{I,i}|$ and the BB term would diverge. At a gas parameter of $na_{BB}^3 = 10^{-6}$, we employ a Gaussian potential $V^{IB}(r) \sim \exp(-r^2/\sigma^2)$ of range $\sigma = 0.01 n^{-1/3}$. Since then $\sigma = a_{BB}$

----

[1] Indeed, one easily checks that this formula fulfils

$$\left(-\frac{\Delta}{2m_B} + V^{IB}(x - x_{I,1}) + V^{IB}(x - x_{I,2}) \right) \phi(x) = 0$$

where $V^{IB}(|r|)\phi(r) = \frac{4\pi a_{IB}}{2m_B} \delta^3(r) \frac{\partial}{\partial|r|} |r|\phi(|r|e_r)$ is the contact potential of scattering length a_{IB}.

is not small compared to a_{BB}, this is, in principle, not soft enough for GPT to be applicable and indeed, we observe differences in energy compared to NLGPT with the same parameters. However, this error is almost constant in d, such that GPT yields the correct result for the induced potential (2) for large separations even for this rather small value of σ.

NLGPT. For the case of strong local variation of n, we have shown in [9] how to extend GPT to a non-local theory by not integrating out the local BB correlations. For two static impurities, the resulting energy functional reads

$$E_{\mathrm{NLGP}} = \int_x \overline{\phi_x}\left(-\frac{\Delta}{2m_{\mathrm{B}}} + V_{x-x_{\mathrm{I},1}}^{\mathrm{IB}} + V_{x-x_{\mathrm{I},2}}^{\mathrm{IB}}\right)\phi_x$$
$$+ \int_{x,y} |\phi_x|^2 |\phi_y|^2 g_{x-y}. \quad (5)$$

Here, $g = \frac{1}{2}\left[\frac{|\nabla f|^2}{m_{\mathrm{B}}} + V^{\mathrm{BB}}|f|^2\right]$ where f is the zero-energy solution to the two-boson problem: $\left(-\frac{\Delta}{m_{\mathrm{B}}} + V^{\mathrm{BB}}\right)f = 0$. We will employ a hard sphere potential, for which $f(x) = \max(0, 1 - a_{\mathrm{BB}}/|x|)$. Note that g is an approximate delta function of range a_{BB} and height $\int g = 4\pi a_{\mathrm{BB}}/2m_{\mathrm{B}}$ and if ϕ is almost constant on these scales, the GP functional is recovered. We compute individual data points by minimizing E_{NLGP} with the same IB interaction potential V^{IB} as for GPT to assert the validity of the latter at large distances as well as to obtain additional results for smaller distances. Still, we limit ourselves to $d \gtrsim \sigma$ where the resulting scaling of V_{ind} is universal.

III. RESULTS

A. Scaling behaviour of the induced potential

Minimizing the GP and NLGP energy functionals at different impurity separations d, we obtain the induced potential shown in Fig. 2. It exhibits the following scaling behaviour:

(a) Large-distance behaviour: Yukawa regime. At large distances, the induced potential falls off like a Yukawa potential [17, 21]

$$V_{\mathrm{ind}}(d) \sim \frac{e^{-d\sqrt{2}/\xi}}{d}.$$

In GPT, this can be seen as follows: The energy of a single impurity in a flat BEC is proportional to n. At large impurity separation, each impurity feels an almost flat density modulation caused by the other impurity, so we should have $E_d \sim n(d)$ where $n(d)$ is the density profile of a BEC with one impurity at zero while n_0 denotes the unperturbed density. $n(d)$ is obtained from the zero-energy GPE with one impurity: At distances larger than the potential
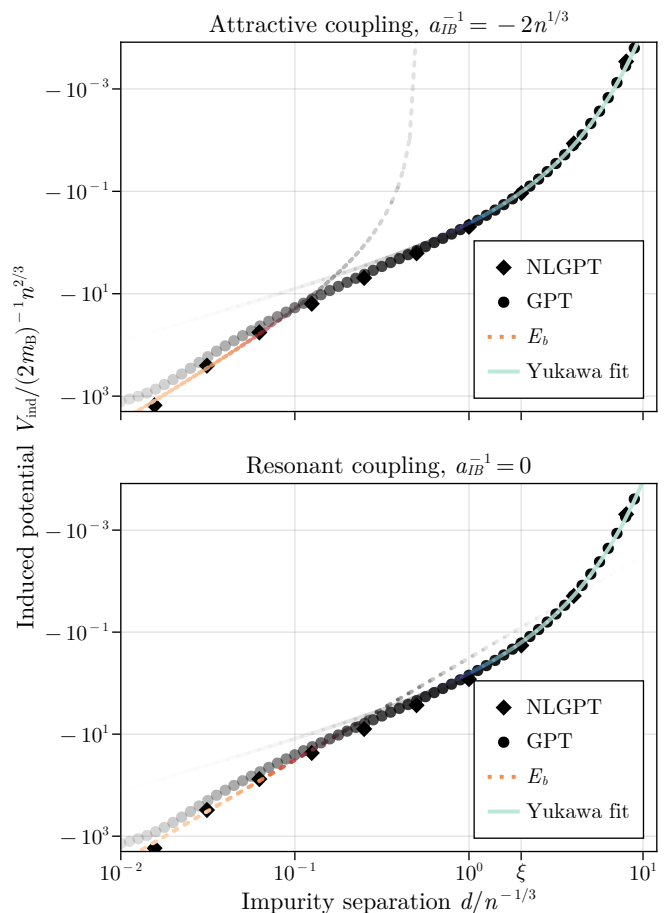

FIG. 2. Induced potential at a gas parameter of $na_{\mathrm{BB}}^3 = 10^{-6}$ and two different coupling strengths (more can be found in the appendix). The circles are the results of the GP simulation and have a Yukawa scaling at large impurity separation. The non-local GPT (diamonds) confirms the applicability of GPT in this regime and remains valid at shorter separations, where GPT results are off by a factor of about 1.5. Here, the three-body binding energy E_{b} is approached, which exhibits Efimov scaling $\sim d^{-2}$ for $d \ll |a_{\mathrm{IB}}|$.

range, it reads (see also [35])

$$0 = -\frac{\Delta}{2m_{\mathrm{B}}}\phi + \frac{8\pi a_{\mathrm{BB}}}{2m_{\mathrm{B}}}\phi(\phi^2 - n_0)$$
$$\Rightarrow \quad 0 = -\delta\phi'' - \frac{2}{d}\delta\phi' + \frac{2}{\xi^2}\delta\phi + \mathscr{O}(\delta\phi^2) \quad (6)$$

where we have set $\phi(d) = \sqrt{n}_0 + \delta\phi(d)$. This is solved by $\delta\phi \sim e^{-d\sqrt{2}/\xi}/d$ and $V_{\mathrm{ind}}(d) \sim n(d) - n_0 \simeq 2\delta\phi$. The linear approximation in $\delta\phi$ is valid when $\delta\phi$ is small, which confirms that the length scale that delimits the Yukawa regime is ξ. Note that this argument does not require GPT to be valid close to an impurity, as it is only used to determine the density profile far away from one impurity.

It must be noted though that the assumption of a finite-range impurity-boson potential was crucial

for the derivation: if V^{IB} has power-law tails, an additional term $2m_{\mathrm{B}}V^{\mathrm{IB}}\sqrt{n}$ appears in (6). At radii large compared to ξ, the asymptotic solution is then $\delta\phi \simeq -\xi^2 m_{\mathrm{B}}\sqrt{n}V^{\mathrm{IB}}$, such that $V_{\mathrm{ind}}(d) \sim V^{\mathrm{IB}}(d)$. This effect can bee observed in Ref. [22] where ions in a BEC interact via an r^{-4} potential, resulting in an r^{-4} tail in the induced interaction[2]. Also, since the analysis is based on Gross-Pitaevskii theory, only the leading order in the gas parameter is captured. At distances large compared to the healing length, sub-leading terms eventually become dominant as the Yukawa potential vanishes [20].

(b) Short-distance behaviour: Efimov regime. At small distances, the induced potential approaches the three-body binding energy E_{b}. This indicates that a single Boson gets bound to the impurity and blocks the bound state for the others due to its repulsion as described in [13]. When d is small against a_{IB}, E_{b} and thus also V_{ind} scale like d^{-2} as in the Efimov effect. At resonance (second panel in Fig. 2), E_{b} is directly proportional to d^{-2} even for larger distances. Nonetheless, we find that the Efimov regime in the many-body system has a finite range of the order of the mean particle distance. This is natural since the picture of one molecule surrounded by a bath breaks down as the size of the Efimov trimer increases and extends into the bath [34].

We are not interested here in even shorter ranges of $d < \sigma$, where the scaling of the potential becomes non-universal and starts to depend on the shape of V^{IB}.

Comparing to the results of [17], we thus also find regions of Yukawa and Efimov scaling, but they are not delimited by the scattering length alone. Instead, we always find both regimes but for different impurity separations while the scattering length controls the size of these regimes. A sketch of this is shown in Fig. 1. The length scale that delimits the Efimov regime is, for weak coupling, the scattering length a_{IB} [16], since the three-body binding energy scales like d^{-2} only for $d \ll |a_{\mathrm{IB}}|$, see (3). If the impurity separation becomes comparable to the mean boson distance $\sim n^{-1/3}$, the picture of an individual boson mediating the interaction no longer applies. Thus, $d \ll n^{-1/3}$ is another condition delimiting the Efimov region and it becomes the dominant one near the resonance. The Yukawa regime, on the other hand, starts when the local density variation $\delta\phi$ from the treatment above becomes sufficiently small for the linear approximation in (6) to be applicable. This turns out to be the case when d is of the order of the healing length ξ.

We note that when d is large compared to ξ, [20] finds that the quickly decaying Yukawa potential becomes domi-

---

[2] The above analysis with the weak-coupling single-impurity energy $E(n) = 4\pi a_{\mathrm{IB}}n/2m_{\mathrm{B}}$ actually leads to a slightly different prefactor $V_{\mathrm{ind}}(d) = -(a_{\mathrm{IB}}/a_{\mathrm{BB}})V^{\mathrm{IB}}(d)$ than in Ref. [22], where a_{IB} is replaced by its Born approximation.
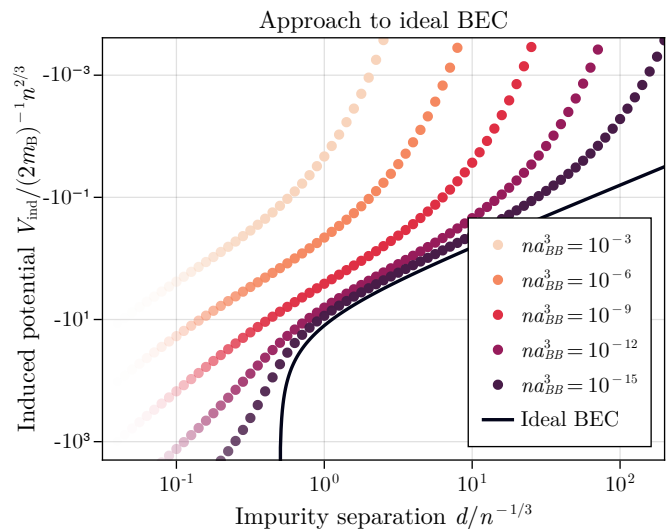

FIG. 3. Induced potential obtained with GPT for varying gas parameters at fixed impurity coupling $a_{\mathrm{IB}} = -0.5n^{-1/3}$. The analytical curve for the ideal BEC exhibiting Newtonian behaviour is approached at very low gas parameters in an intermediate distance regime $|a_{\mathrm{IB}}| < d < \xi$.

nated by second-order effects, which lead to van-der-Waals scaling.

B. Approach to ideal BEC

In the beginning, we have shown that attractively coupled impurities in an ideal BEC lead to a shifted Newtonian potential. This matches neither the Efimov nor the Yukawa potential found for the real BEC, so one may wonder how the Newtonian regime emerges as an ideal BEC is approached. In Fig. 3, we solve the GPE at fixed impurity coupling a_{IB} for a sequence of BEC gas parameters na_{BB}^3. As it is lowered, the ideal gas result is approached in an intermediate distance regime, which is bounded by $|a_{\mathrm{IB}}|$ from below hand and ξ from above. However, the approach is very slow and gas parameters of the order of 10^{-15} are required for it to be visible. This is a consequence of the slow splitting of length scales in a BEC: the healing length in units of $n^{-1/3}$ scales with order $-1/6$ in the gas parameter, so very small values of the latter are required to push the Yukawa regime to sufficiently large distances to make room for an intermediate Newtonian scaling regime.

IV. CONCLUSION

We have investigated the problem of a force between particles induced by an interacting Bose-Einstein condensed medium by minimizing local and non-local Gross-Pitaevskii energy functionals. We found that both the Efimov and Yukawa scaling of the mediated potential can

be observed in certain distance regions with an intermediate cross-over regime (Fig. 1). This is in contrast to previous studies predicting either one or the other depending on the coupling strength. Our result, valid for impurity separations up to the order of the healing length, complements a recent study treating separations larger than the healing length [20]. A non-interacting condensate, on the other hand, was shown to induce a shifted Newtonian potential. The cross-over from interacting to non-interacting condensate was found to take place at low gas parameters of order 10^{-15}, where the Newtonian behaviour emerges in an intermediate-distance regime. Experimentally, induced interactions are expected to lead to an energy shift depending on impurity concentration, which may be probed by radio-frequency spectroscopy [1–4]. To investigate the distance dependence, the impurities can be confined to individual microtraps [22]. For future work, it will be interesting to investigate the influence of finite impurity mass in a region where it is still large compared to the Boson mass, such that the picture of mediated forces can be applied.

Appendix A: Induced potential for more values of the coupling

In Fig. 4, we show the induced potentials as in Fig. 2 for more values of the coupling a_{IB}, including the repulsive side of a resonance. The scaling regimes observed from these curves lead to Fig. 1.

[1] M.-G. Hu, M. J. Van de Graaff, D. Kedar, J. P. Corson, E. A. Cornell, and D. S. Jin, Bose Polarons in the Strongly Interacting Regime, Physical Review Letters **117**, 055301 (2016).

[2] N. B. Jørgensen, L. Wacker, K. T. Skalmstang, M. M. Parish, J. Levinsen, R. S. Christensen, G. M. Bruun, and J. J. Arlt, Observation of Attractive and Repulsive Polarons in a Bose-Einstein Condensate, Physical Review Letters **117**, 055302 (2016).

[3] Z. Z. Yan, Y. Ni, C. Robens, and M. W. Zwierlein, Bose polarons near quantum criticality, Science **368**, 190 (2020).

[4] M. G. Skou, T. G. Skov, N. B. Jørgensen, K. K. Nielsen, A. Camacho-Guardian, T. Pohl, G. M. Bruun, and J. J. Arlt, Non-equilibrium quantum dynamics and formation of the Bose polaron, Nature Physics , 1 (2021).

[5] S. P. Rath and R. Schmidt, Field-theoretical study of the Bose polaron, Physical Review A **88**, 053632 (2013).

[6] Y. E. Shchadilova, F. Grusdt, A. N. Rubtsov, and E. Demler, Polaronic mass renormalization of impurities in Bose-Einstein condensates: Correlated Gaussian-wave-function approach, Physical Review A **93**, 043606 (2016).

[7] J. Levinsen, M. M. Parish, R. S. Christensen, J. J. Arlt, and G. M. Bruun, Finite-temperature behavior of the Bose polaron, Physical Review A **96**, 063622 (2017).

[8] M. Drescher, M. Salmhofer, and T. Enss, Real-space dynamics of attractive and repulsive polarons in Bose-Einstein condensates, Physical Review A **99**, 023601 (2019).

[9] M. Drescher, M. Salmhofer, and T. Enss, Theory of a resonantly interacting impurity in a Bose-Einstein condensate, Physical Review Research **2**, 032011(R) (2020).

[10] D. Dzsotjan, R. Schmidt, and M. Fleischhauer, Dynamical Variational Approach to Bose Polarons at Finite Temperatures, Physical Review Letters **124**, 223401 (2020).

[11] R. Schmidt and T. Enss, Self-stabilized Bose polarons, SciPost Physics **13**, 054 (2022).

[12] M. Drescher, M. Salmhofer, and T. Enss, Quench Dynamics of the Ideal Bose Polaron at Zero and Nonzero Temperatures, Physical Review A **103**, 033317 (2021).

[13] J. Levinsen, L. A. P. Ardila, S. M. Yoshida, and M. M. Parish, Quantum Behavior of a Heavy Impurity Strongly Coupled to a Bose Gas, Physical Review Letters **127**, 033401 (2021).

[14] F. Rose and R. Schmidt, Disorder in order: Localization without randomness in a cold-atom system, Physical Review A **105**, 013324 (2022).

[15] T. Enss, N. Cuadra Braatz, and G. Gori, Complex scaling flows in the quench dynamics of interacting particles, Physical Review A **106**, 013308 (2022), arXiv:2203.06098.

[16] N. T. Zinner, Efimov states of heavy impurities in a Bose-Einstein condensate, EPL (Europhysics Letters) **101**, 60009 (2013).

[17] P. Naidon, Two Impurities in a Bose–Einstein Condensate: From Yukawa to Efimov Attracted Polarons, Journal of the Physical Society of Japan **87**, 043002 (2018).

[18] A. Camacho-Guardian and G. M. Bruun, Landau Effective Interaction between Quasiparticles in a Bose-Einstein Condensate, Physical Review X **8**, 031042 (2018).

[19] G. Panochko and V. Pastukhov, Two- and three-body effective potentials between impurities in ideal BEC, Journal of Physics A: Mathematical and Theoretical **54**, 085001 (2021).

[20] K. Fujii, M. Hongo, and T. Enss, Universal van der Waals Force between Heavy Polarons in Superfluids, Physical Review Letters **129**, 233401 (2022).

[21] J. Jager and R. Barnett, The effect of boson–boson interaction on the bipolaron formation, New Journal of Physics **24**, 103032 (2022).

[22] S. Ding, M. Drewsen, J. J. Arlt, and G. M. Bruun, Mediated Interaction between Ions in Quantum Degenerate Gases, Physical Review Letters **129**, 153401 (2022).

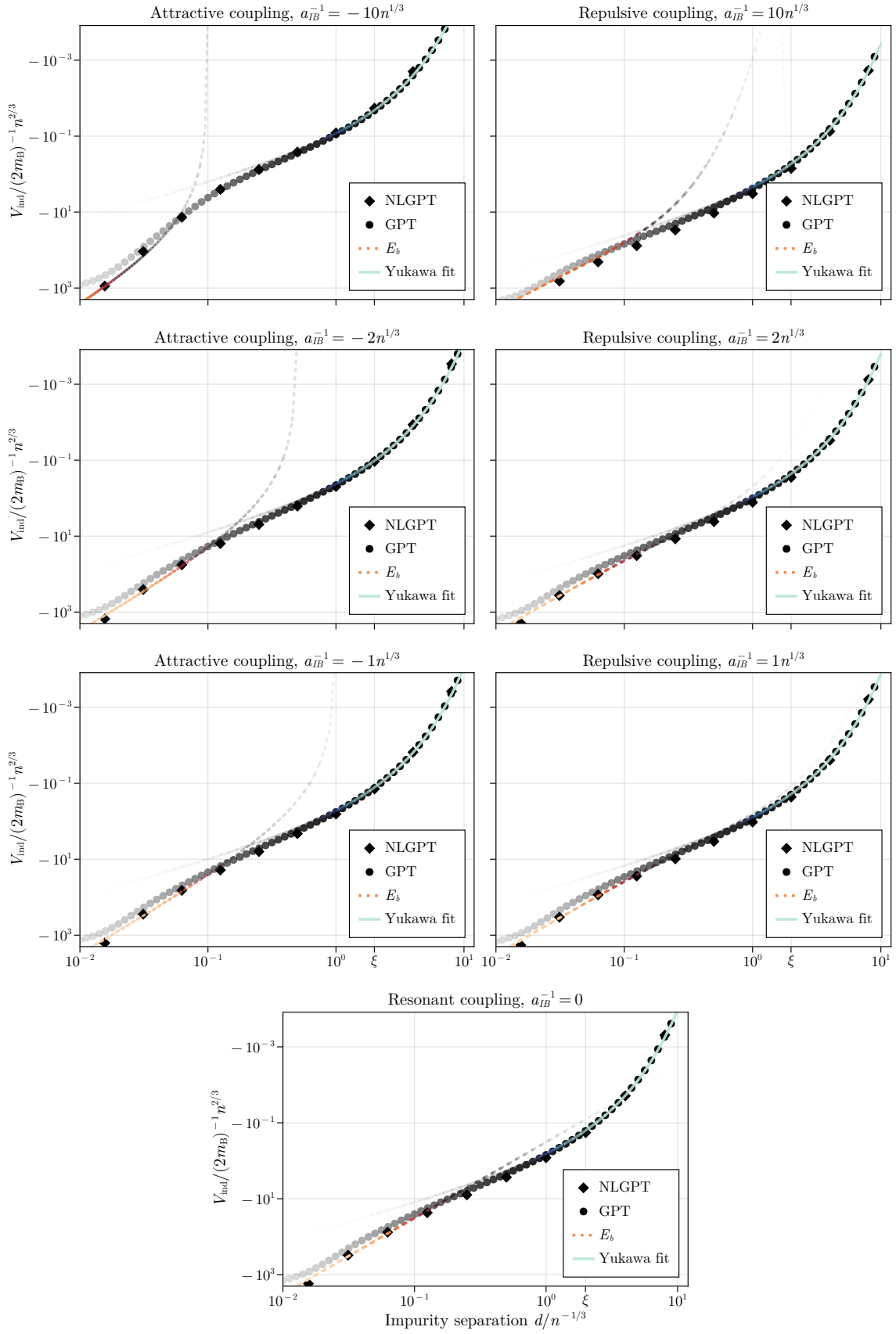[23] G. Bighin, P. A. Murthy, N. Defenu, and T. Enss, Resonantly enhanced superconductivity mediated by spinor

FIG. 4. Induced potential for various coupling strengths a_{IB}. The gas parameter is $na_{\mathrm{BB}}^3 = 10^{-6}$ as in Fig. 2.

condensates (2022), arXiv:2212.07419 [cond-mat].

[24] M. Schecter and A. Kamenev, Phonon-Mediated Casimir Interaction between Mobile Impurities in One-Dimensional Quantum Liquids, Physical Review Letters **112**, 155301 (2014).

[25] A. S. Dehkharghani, A. G. Volosniev, and N. T. Zinner, Coalescence of Two Impurities in a Trapped One-dimensional Bose Gas, Physical Review Letters **121**, 080405 (2018).

[26] S. I. Mistakidis, A. G. Volosniev, and P. Schmelcher, Induced correlations between impurities in a one-dimensional quenched Bose gas, Physical Review Research **2**, 023154 (2020).

[27] M. Will, G. E. Astrakharchik, and M. Fleischhauer, Polaron Interactions and Bipolarons in One-Dimensional Bose Gases in the Strong Coupling Regime, Physical Review Letters **127**, 103401 (2021).

[28] A. Petković and Z. Ristivojevic, Mediated interaction between polarons in a one-dimensional Bose gas, Physical Review A **105**, L021303 (2022).

[29] Y. Nishida, Casimir interaction among heavy fermions in the BCS-BEC crossover, Physical Review A **79**, 013629 (2009).

[30] B. J. DeSalvo, K. Patel, G. Cai, and C. Chin, Observation of fermion-mediated interactions between bosonic atoms, Nature **568**, 61 (2019).

[31] H. Edri, B. Raz, N. Matzliah, N. Davidson, and R. Ozeri, Observation of Spin-Spin Fermion-Mediated Interactions between Ultracold Bosons, Physical Review Letters **124**, 163401 (2020).

[32] T. Enss, B. Tran, M. Rautenberg, M. Gerken, E. Lippi, M. Drescher, B. Zhu, M. Weidemüller, and M. Salmhofer, Scattering of two heavy Fermi polarons: Resonances and quasibound states, Physical Review A **102**, 063321 (2020).

[33] B. Tran, M. Rautenberg, M. Gerken, E. Lippi, B. Zhu, J. Ulmanis, M. Drescher, M. Salmhofer, T. Enss, and M. Weidemüller, Fermions Meet Two Bosons—the Heteronuclear Efimov Effect Revisited, Brazilian Journal of Physics **51**, 316 (2021).

[34] M. Sun, H. Zhai, and X. Cui, Visualizing the Efimov Correlation in Bose Polarons, Physical Review Letters **119**, 013401 (2017).

[35] P. Massignan, N. Yegovtsev, and V. Gurarie, Universal Aspects of a Strongly Interacting Impurity in a Dilute Bose Condensate, Physical Review Letters **126**, 123403 (2021).