Gluon distribution functions from lattice QCD in the light cone limit

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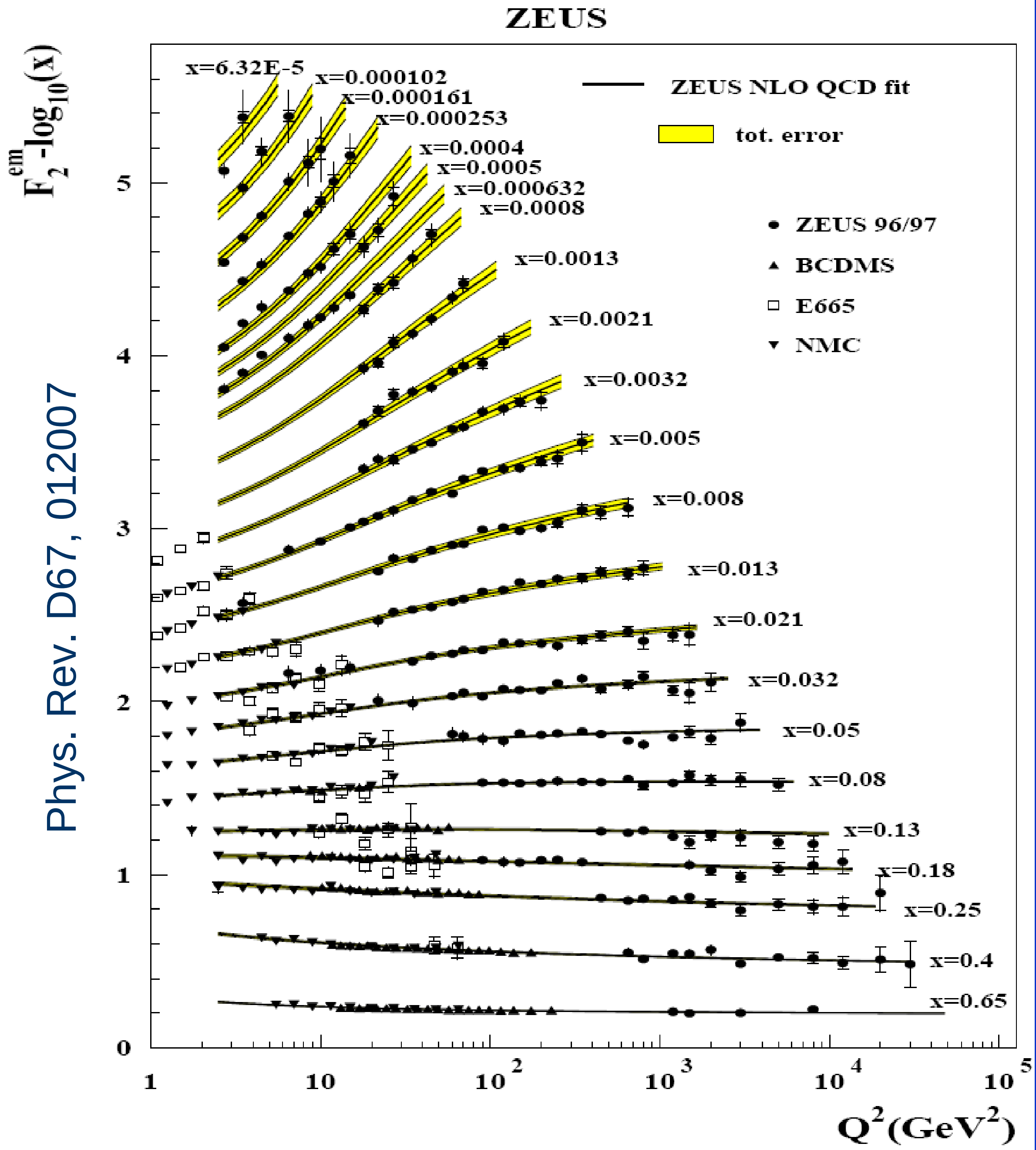
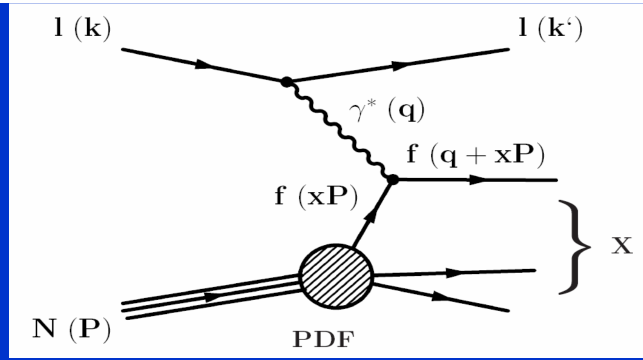
Outline

- Review and Motivation
- Near-Light-Cone coordinates
- NLC lattice ground state
- Construction of the Dipole state
- Construction of the gluon distribution function on the lattice
- Discussion of the results for a one link dipole
- Discussion of the gluon distribution function of a hadron
- Conclusions

Phys. Rev. D67, 012007





- QCD factorization theorem tells us separation of hard from soft physics
- Evolve u,d valence,g and total sea quark distributions by DGLAP

$$xf(x) = p_1 x^{p_2}(1-x)^{p_3}(1+p_5 x)$$

- Fit to data
- Excellent agreement Success story for perturbative QCD
- However: This is an ansatz. Is there a possibility to compute the structure function at some input scale directly?

Gluon distribution function

Collins and Soper, Nucl. Phys. B194, 445

$$g(x_B) = \frac{1}{x_B}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} dz^-\, d^2\vec{z}_\perp\, e^{-\mathrm{i}\, x_B\, p_-\, z^-}\,\frac{1}{p_-}\left\langle h(p_-,\vec{0}_\perp)\right| G(z^-,\vec{z}_\perp\,;0,\vec{z}_\perp) \left| h(p_-,\vec{0}_\perp)\right\rangle_c \Big|_{z^+=0}$$

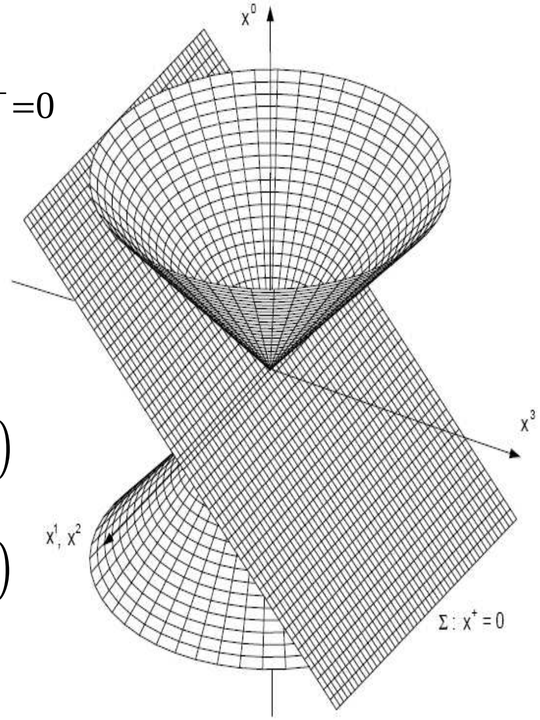
$$G(z^-,\vec{z}_\perp\,;\,0,\vec{z}_\perp) = \sum_{k=1}^{2} F^a_{-k}(z^-,\vec{z}_\perp)\, S^A_{ab}(z^-,\vec{z}_\perp\,;\,0,\vec{z}_\perp)\, F^b_{-k}(0,\vec{z}_\perp)$$

$$S^A_{ab}(z^-,\vec{z}_\perp\,;\,0,\vec{z}_\perp) = \left[\mathcal{P}\exp\left\{\mathrm{i}\, g \int_0^{z^-} dx^-\, A^c_-(x^-,\vec{z}_\perp)\,\lambda^c_{adj}\right\}\right]_{ab}$$

$$z^+ = \frac{1}{\sqrt{2}}\left(z^0 + z^3\right)$$

$$z^- = \frac{1}{\sqrt{2}}\left(z^0 - z^3\right)$$

$$G(z^-,\vec{z}_\perp\,;\,0,\vec{z}_\perp)\big|_{z^-=0} = \mathcal{P}^{lc}_-(0,\vec{z}_\perp) = \sum_{k=1}^{2} F^a_{-k}(0,\vec{z}_\perp)\, F^a_{-k}(0,\vec{z}_\perp)$$



- Hadron is probed at equal light cone time
  $\Longrightarrow$ Static problem in light cone quantization

Motivation

- Structure functions at input scale not computable perturbatively (manifestly non-perturbative) $\Longrightarrow$ lattice methods
- Euclidean equal time lattice methods capable of computing moments by OPE (Martinelli and Sachrajda Nucl. Phys. B 306,865)
- Light cone quantisation seems to be natural to describe high energy scattering
- Is there a way to combine light cone quantization with lattice methods ?
  - Yes: transverse lattice method (Bardeen et al. Phys. Rev. D21,1037)

    Basis Light Front Quantization Approach (Vary et al.)
  - Yes: Lattice QCD near the light cone (Wilsonian approach)

    (D.G, E.-M. I., H.-J. P. and E.P.: Phys.Rev.D77:014512,2008)

Near light-cone coordinates

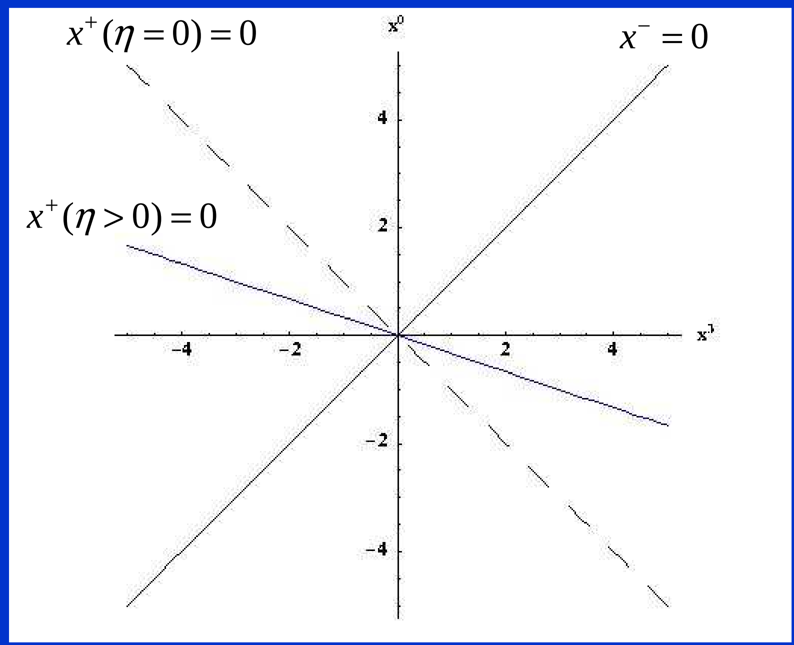
Prokhvatilov et. al, Sov. J. of Nucl. Phys.49 (688); Lenz et. al, Annals of Physics 208 (1-89)

- Transition to NLC coordinates is a two step process
  - Lorentz boost to a fast moving frame with relative velocity

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^3) \\ x'^3 &= \gamma(x^3 - \beta x^0) \end{aligned} \qquad \beta = \frac{1-\eta^2/2}{1+\eta^2/2}$$

  - Rotation in the x'^0-x'^3-plane

$$\begin{aligned} x^+ &= \frac{1}{\sqrt{2}}\left[\left(1+\frac{\eta^2}{2}\right)x'^0 + \left(1-\frac{\eta^2}{2}\right)x'^3\right] \\ x^- &= \frac{1}{\sqrt{2}}\left[x'^0 - x'^3\right] \end{aligned}$$



- Allows interpolation between equal-time $\eta^2 = 2$ and light-cone quantization $\eta^2 = 0$
- Introduced to investigate light-cone quantization as a limiting procedure of equal time theories

$$H = E_-^2 + B_-^2 + \frac{1}{\eta^2}(E_i - B_i)^2$$

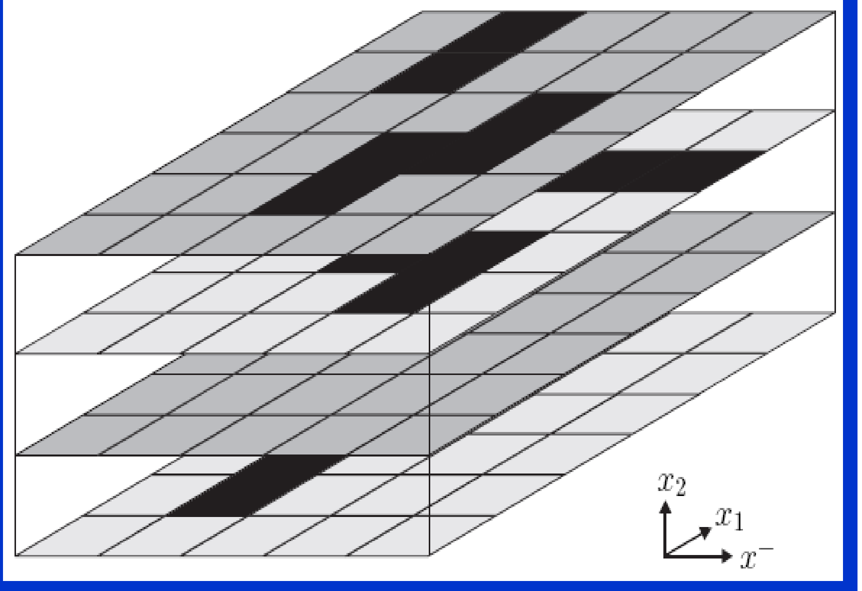
Lattice Ground state in the light cone limit

$$|\Psi_0\rangle = \Psi_0[U]\ |0\rangle = \sqrt{N_\Psi}\, e^{f[U]}\ |0\rangle\ ,$$

$$f[U] = \sum_{\vec{x}} \left\{ \sum_{k=1}^{2} \rho_0(\lambda,\tilde{\eta})\, \mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right] + \delta_0(\lambda,\tilde{\eta})\, \mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right] \right\} |0\rangle$$

$$\rho_0(\lambda,0) = \left(0.65 - \frac{0.87}{\lambda} + \frac{1.65}{\lambda^2}\right)\sqrt{\lambda}\,, \qquad \rho_0(10,0) = 1.83$$

$$\delta_0(\lambda,0) = \left(0.05 + \frac{0.04}{\lambda} - \frac{1.39}{\lambda^2}\right)\sqrt{\lambda}\,. \qquad \delta_0(10,0) = 0.13$$



- Essentially ensemble of decoupled 2-d theories
- SIMPLIFICATION: Strong coupling methods become exact

$$\langle\Psi_0|\,\frac{1}{2}\,\mathrm{Tr}\left[W(0,\vec{0}_\perp\,;\,z^-,\vec{d}_\perp)\right]|\Psi_0\rangle = \left(\langle\Psi_0|\,\frac{1}{2}\mathrm{Tr}\,[\,U_{-k}\,]\,|\Psi_0\rangle\right)^{d_\perp|z^-|}$$

$$f_{1k} \equiv \langle\Psi_0|\,\frac{1}{2}\mathrm{Tr}\,[\,U_{-k}\,]\,|\Psi_0\rangle = \frac{I_2(4\,\rho_0)}{I_1(4\,\rho_0)} + \mathcal{O}(\delta_0^2) \in [-1,1] \qquad f_{2k} \equiv \langle\Psi_0|\left(\frac{1}{2}\mathrm{Tr}\,[\,U_{-k}\,]\right)^2|\Psi_0\rangle = \frac{I_2(4\,\rho_0)}{4\,\rho_0\, I_1(4\,\rho_0)} + \frac{I_3(4\,\rho_0)}{I_1(4\,\rho_0)} + \mathcal{O}(\delta_0^2) \in [0,1]$$

Color-Dipole Model

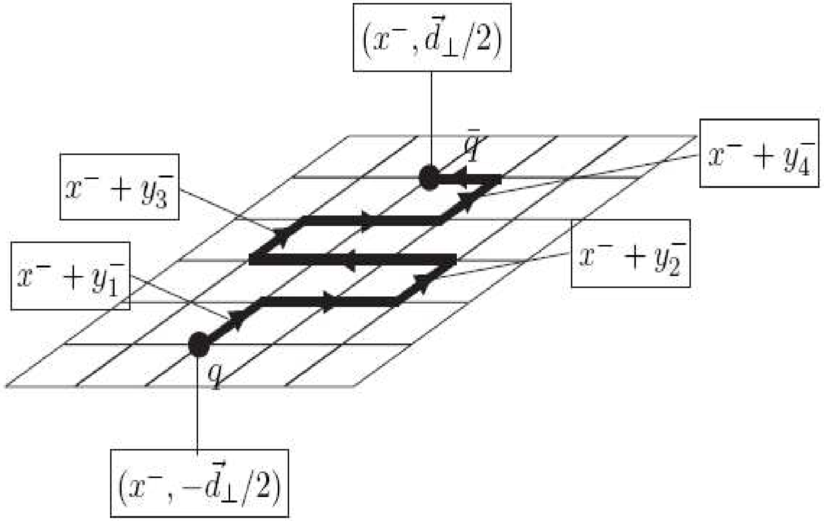
- Consider scalar QCD $\Phi(x) = \Phi_+(x) + \Phi_-(x)$

- Interpolating color-dipole field of two heavy quarks with separation $\vec{d}$ in config. space:

  $\Longrightarrow$ Gluons represented by Schwinger string

- Equip with total momentum $(p_-, \vec{p}_\perp) = (p_-, \vec{0}_\perp)$

$$\left| d(p_-; -\vec{d}_\perp/2, \mathrm{C}_\perp, \vec{d}_\perp/2) \right\rangle$$



$$\Psi(\{y_j^-\}) = \int \left(\prod_{j=1}^{n} dl_-^j \right) e^{-\mathrm{i} \sum_{j=1}^{n} l_-^j y_j^-} \Theta\left(p_- - \sum_{j=1}^{n} l_-^j \right)$$

NLC correlation function on the lattice

- NLC lattice correlation function as a point split generalization of the longitudinal lattice momentum operator

$$\left[P_-, \uparrow_{\vec{y}}\right] = \frac{1}{2\mathrm{i}}\left(\sqsupset_{\vec{y}+\vec{e}_-} - \sqsubset_{\vec{y}-\vec{e}_-} \right) + \mathcal{O}(a^2)$$

$$\left[\sum_{\vec{z}} \mathcal{P}_-(\vec{z}), V_j(p_-, \vec{y}_\perp)\right] = \sin(p_-)\, V_j(p_-, \vec{y}_\perp) + \mathcal{O}(a^2)$$

$$p_- = \frac{2\pi}{N\_a\_} n \in [0,\pi]$$

$$V_j(p_-, \vec{y}_\perp) = \sum_{y^-} e^{-\mathrm{i}p_- y^-}\, U_j(y^-, \vec{y}_\perp)$$

- PROBLEM:
  - Spectrum is not unique
  - Largest lattice momentum corresponds to an eigenvalue of the momentum operator approximately equal to zero

- Introduce a block averaged momentum eigen state on the original lattice by blocking over fine lattice links

$$\widetilde{V}_j(p_-, \vec{y}_\perp) = \sum_{y_f^-} e^{-\mathrm{i}p_-^f y_f^-}\, U_j^f(y_f^-, \vec{y}_\perp)\Big|_{p_-^f = p_-/2}$$

$$L_{\mathrm{phys}} = N_- a_- \qquad N_- = \frac{N_-^f}{2}$$

$$a_- = 2\,a_-^f \qquad p_-^f = \frac{p_-}{2}$$

$$\left[\widetilde{P}_-, \widetilde{V}_j(p_-, \vec{y}_\perp)\right] = 2\,\sin(p_-/2)\,\widetilde{V}_j(p_-, \vec{y}_\perp)$$

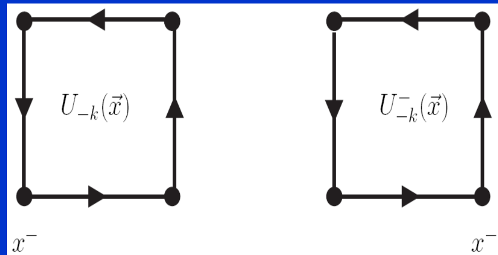
- Introduce block averaged „effective“ longitudinal momentum operator

$$\widetilde{\mathcal{P}}_-(z^-,\vec z_\perp) = 2\left(\frac{1}{2}\mathcal{P}^f_-(2\,z^- - 1,\vec z_\perp) + \mathcal{P}^f_-(2\,z^-,\vec z_\perp) + \frac{1}{2}\mathcal{P}^f_-(2\,z^- + 1,\vec z_\perp)\right)$$

- Point split generalization of the effective longitudinal momentum operator

$$\widetilde{G}(z^-,\vec z_\perp\,;\,z^{-\prime},\vec z_\perp) \;=\; 2\,\Bigg(\frac{1}{2}G^f(2\,z^- - 1,\vec z_\perp\,;\,2\,z^{-\prime} - 1,\vec z_\perp) + G^f(2\,z^-,\vec z_\perp\,;\,2\,z^{-\prime},\vec z_\perp) + \frac{1}{2}G^f(2\,z^- + 1,\vec z_\perp\,;\,2\,z^{-\prime} + 1,\vec z_\perp)\Bigg)\,.$$

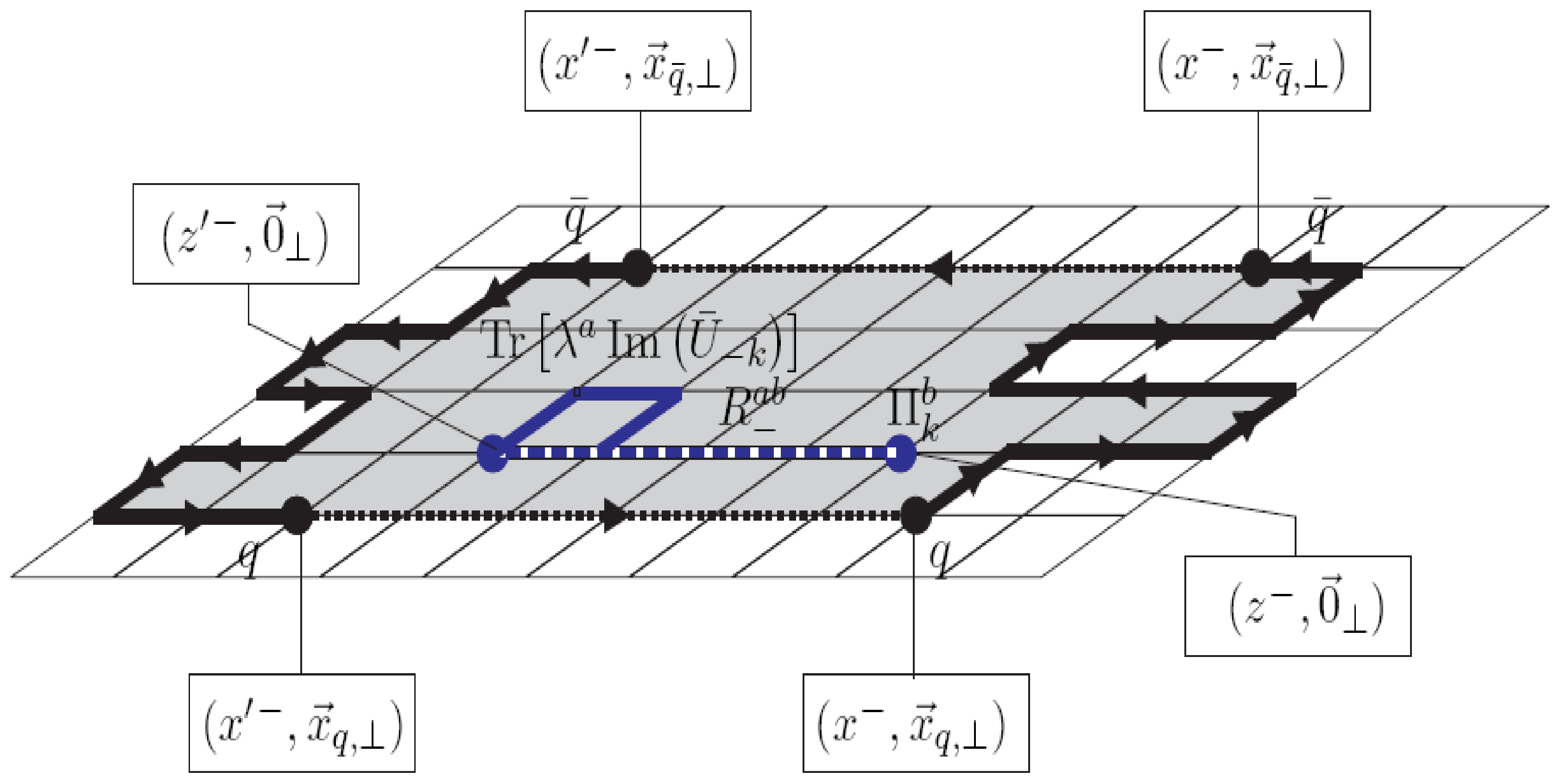
$$a_-^2\,a_\perp^2\,G_{\rm lat}(z^-,z^{\prime -}) = \frac{1}{4}\frac{1}{N_-}\sum_k\Bigg(2\,\Pi^a_k(z^-,\vec 0_\perp)\,S^A_{ab}(z^-,z^{\prime -}\,;\,\vec 0_\perp)\,{\rm Tr}\left[\frac{\sigma^a}{2}\,{\rm Im}\Big(\overline{U}_{-k}(z^{\prime -},\vec 0_\perp)\Big)\right] + 2\,\Pi^a_k(z^{\prime -},\vec 0_\perp)\,S^A_{ab}(z^{\prime -},z^-\,;\,\vec 0_\perp)\,{\rm Tr}\left[\frac{\sigma^a}{2}\,{\rm Im}\Big(\overline{U}_{-k}(z^-,\vec 0_\perp)\Big)\right] + h.c.\Bigg)$$



- Gluon distribution function on the lattice

$$g(p^g_-) = \lim_{\eta\to 0}\frac{2}{p^g_-}\frac{1}{N_-}\sum_{z^-,z^{\prime -}}\sum_{\vec z_\perp} e^{-{\rm i}\,p^g_-(z^- - z^{\prime -})}\,\frac{\Big\langle d(p_-\,;\,-\vec d_\perp,\mathcal{C}_\perp,\vec d_\perp/2)\Big|\,\widetilde{G}(z^-,\vec z_\perp\,;\,z^{\prime -},\vec z_\perp)\,\Big|d(p_-\,;\,-\vec d_\perp,\mathcal{C}_\perp,\vec d_\perp/2)\Big\rangle_c}{\Big\langle d(p_-\,;\,-\vec d_\perp,\mathcal{C}_\perp,\vec d_\perp/2)\,\Big|d(p_-\,;\,-\vec d_\perp,\mathcal{C}_\perp,\vec d_\perp/2)\Big\rangle}$$

Visualization of matrix elements

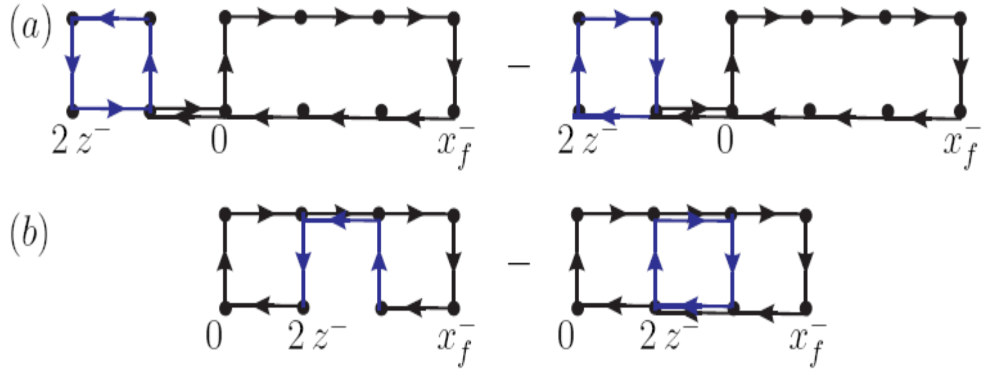


$p^S_- = p_- - p^q_- - p^{\bar{q}}_-$ fixed by experimental input

Computation of matrix elements

$$\widetilde{G}\, S^f_{q\bar{q}}\, |\Psi_0\rangle = \left[\widetilde{G}, S^f_{q\bar{q}}\right] |\Psi_0\rangle + S^f_{q\bar{q}} \left[\widetilde{G}, \Psi_0\right] |0\rangle + S^f_{q\bar{q}}\, \Psi_0\, \widetilde{G}\, |0\rangle$$

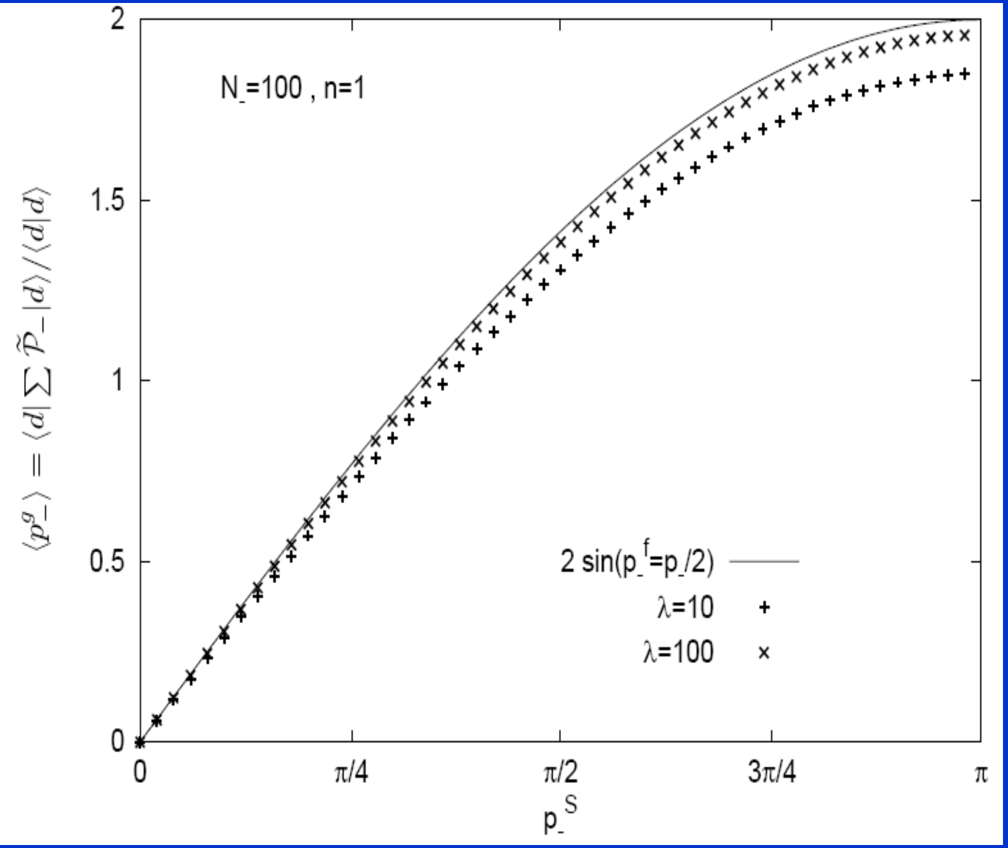
$$\begin{aligned}
&\left[G^f(z_f^-, \vec{z}_\perp\,;\, z_f^{-\prime}, \vec{z}_\perp)\,,\, U^f_j(x_f^-, \vec{x}_\perp)\right] = \\
&\quad \frac{1}{2} S^f_-(x_f^-, \vec{x}_\perp\,;\, z_f^-, \vec{z}_\perp)\,\mathrm{Im}\Big(\bar{U}^f_{-j}(z_f^-, \vec{x}_\perp)\Big)\, S^f_-(z_f^-, \vec{x}_\perp\,;\, x_f^-, \vec{x}_\perp) U^f_j(x_f^-, \vec{x}_\perp)\, \delta_{x_f^-, z_f^{-\prime}}\, \delta_{\vec{x}_\perp, \vec{z}_\perp} \\
&\quad + (z_f^- \leftrightarrow z_f^{-\prime})\,.
\end{aligned} \tag{66}$$



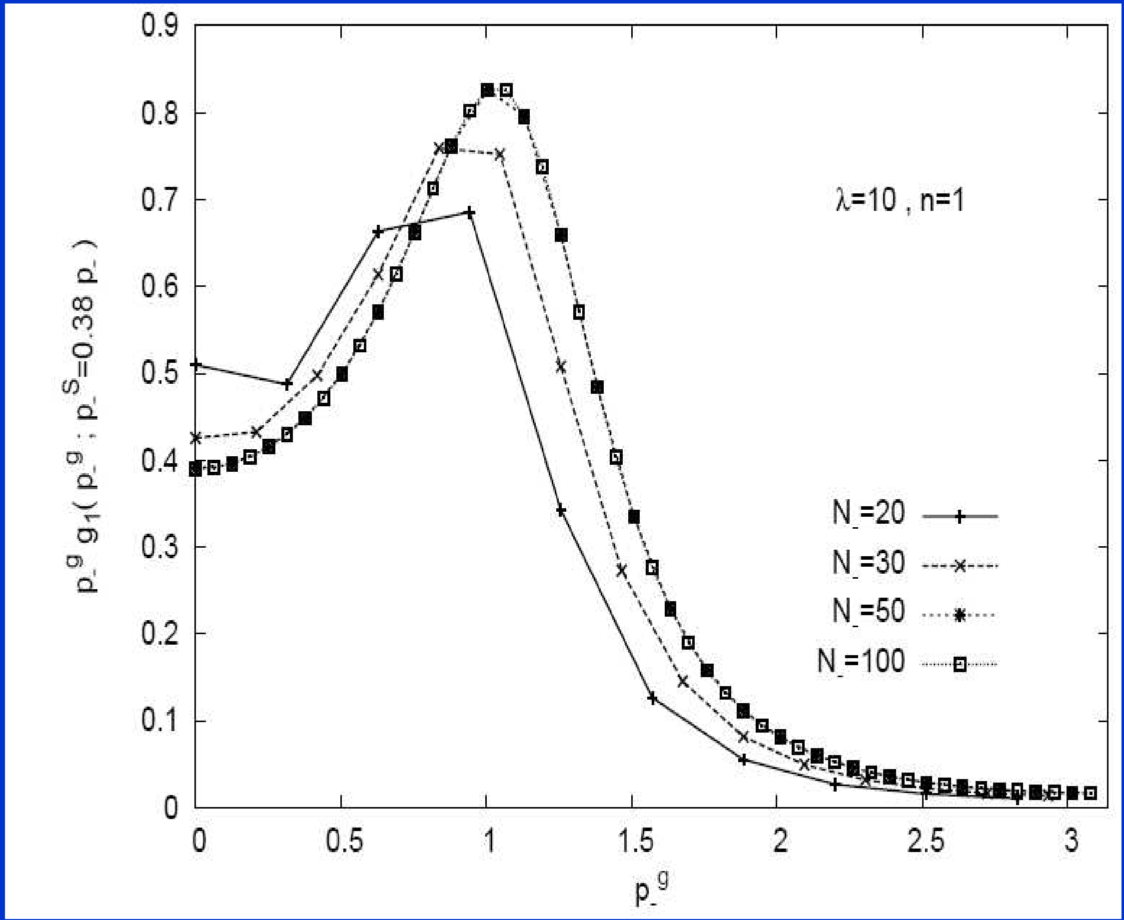
$$\begin{aligned}
g_1(p_-^g, p_-^S) \;=\; & \frac{2}{p_-^g} \sum_{z^-} 2\cos\left(p_-^g\, z^-\right) \frac{1}{F_1(p_-^S)} \sum_{x_f^-} 2\sin\left(\frac{p_-^S}{2}\, x_f^-\right) \\
& \left(1 - \left\langle \left(\frac{1}{2}\mathrm{Tr}\left[U^f_{-k}\right]\right)^2 \right\rangle\right) \left\langle \frac{1}{2}\mathrm{Tr}\left[U^f_{-k}\right] \right\rangle^{|x_f^-|-1} \\
& \cdot \left[\frac{1}{2}\Big(\Theta_0(2\,z^-)\Theta(x_f^- - 2\,z^-) + \Theta(2\,z^-)\Theta_0(x_f^- - 2\,z^-)\Big)\right. \\
& \qquad \left. - \frac{1}{2}\Big(\Theta(-2\,z^-)\Theta_0(2\,z^- - x_f^-) + \Theta_0(-2\,z^-)\Theta(2\,z^- - x_f^-)\Big)\right]
\end{aligned}$$

Momentum sum rule

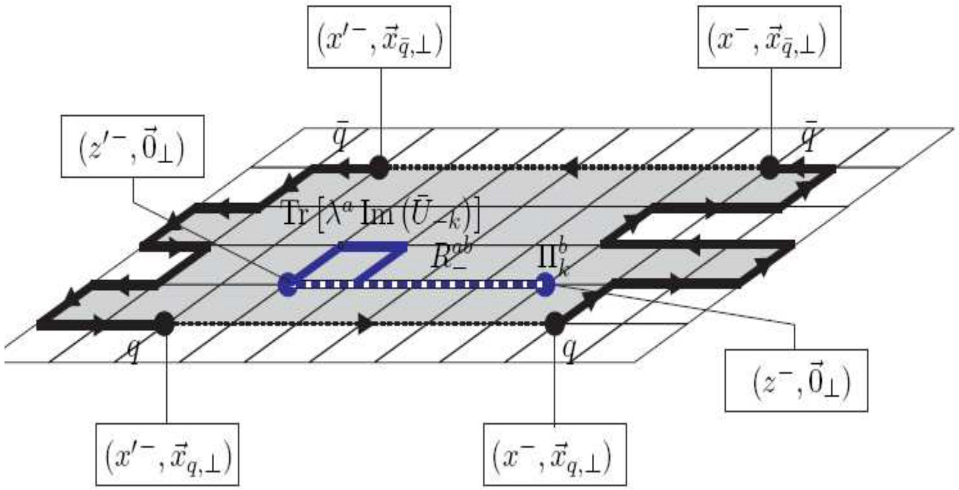
$$\langle p_-^g \rangle = \sum_{p_-^g=0}^{p_-} p_-^g \, g(p_-^g) = \sum_{z^-,\vec{z}_\perp} \left\langle \widetilde{\mathcal{P}}_- \right\rangle + \mathcal{O}\left(\frac{1}{N_-}\right)$$

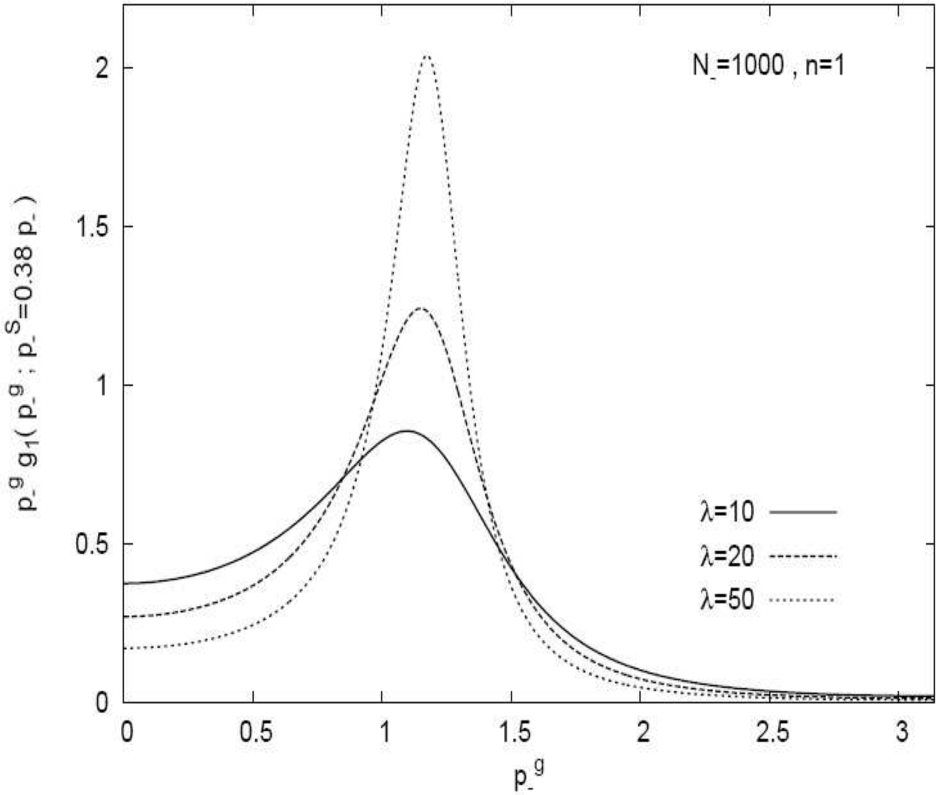


Gluon distribution function as a function of the lattice extension



Gluon distribution as a function of the gauge coupling





$$W(0,\vec{0}_\perp\,;\,x_f{}^-,\vec{d}_\perp) = \left\langle \frac{1}{2}\,\mathrm{Tr}\left[U^f_{-k}\right]\right\rangle^{|x_f^-|} \qquad \Delta\xi = \log(\frac{1}{2})/\log\left(\left\langle\frac{1}{2}\,\mathrm{Tr}\left[U^f_{-k}\right]\right\rangle\right) \qquad \Delta p_-^g = \frac{1}{\Delta\xi}$$

Gluon distribution function of a hadron

- Hadron = superpostition of dipole states

$$\left|h(p_-,\vec{0}_\perp)\right\rangle = \sum_{\mathcal{C},\vec{d}_\perp} \Psi_h(\mathcal{C},\vec{d}_\perp)\left|d(p_-\,;\,-\vec{d}_\perp/2,\mathcal{C}_\perp,\vec{d}_\perp/2)\right\rangle$$

$$\Psi_h(\mathcal{C},\vec{d}_\perp) \equiv \left\langle d(p_-\,;\,-\vec{d}_\perp/2,\mathcal{C}_\perp,\vec{d}_\perp/2)\middle|h(p_-,\vec{0}_\perp)\right\rangle.$$

- Projection on Jz=0 implemented by random walk in the transversal plane

$$\langle R_\perp^2\rangle = \frac{n\,a_\perp^2}{2}$$

- Strong coupling:

$$\left\langle h(p_-,\vec{0}_\perp)\middle| O \middle|h(p_-,\vec{0}_\perp)\right\rangle =$$

$$\sum_{\mathcal{C},\vec{d}_\perp} \left|\Psi_h(\mathcal{C},\vec{d}_\perp)\right|^2 \left\langle d(p_-\,;\,-\vec{d}_\perp/2,\mathcal{C}_\perp,\vec{d}_\perp/2)\middle| O \middle|d(p_-\,;\,-\vec{d}_\perp/2,\mathcal{C}_\perp,\vec{d}_\perp/2)\right\rangle$$

- Hadron distribution function is equivalent to a n link dipole distribution function

$$g_h(p_-^g;p_-^S) = g_n(p_-^g;p_-^S)$$

- DGLAP type of evolution equation (strong coupling equivalent)

$$g_n\left(p_-^g;p_-^S\right) = \sum_{p_-^{S\prime}=0}^{p_-^S} g_{n-1}\left(p_-^g;p_-^{S\prime}\right) P_{n\to n-1}(p_-^S,p_-^{S\prime})$$

$$\sum_{p_-^{S\prime}=0}^{p_-^S} p_-^{S\prime} P_{n\to n-1}(p_-^S,p_-^{S\prime}) = p_-^S$$

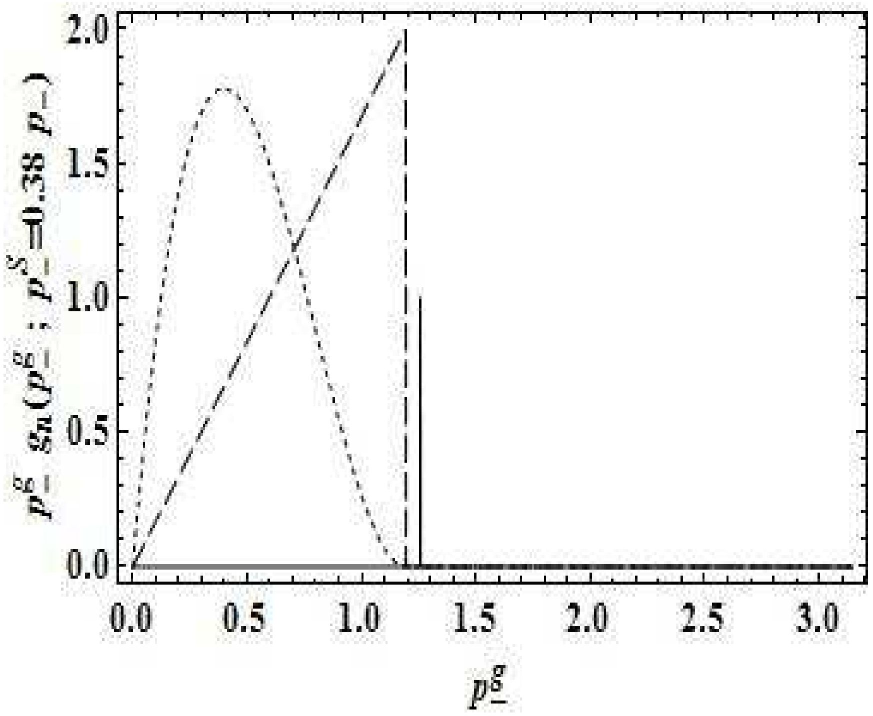
$$\sum_{p_-^g=0}^{p_-} p_-^g\, g_n(p_-^g;p_-^S) = p_-^S$$

- If one has scaling in the limit
  the evolution equation obeys

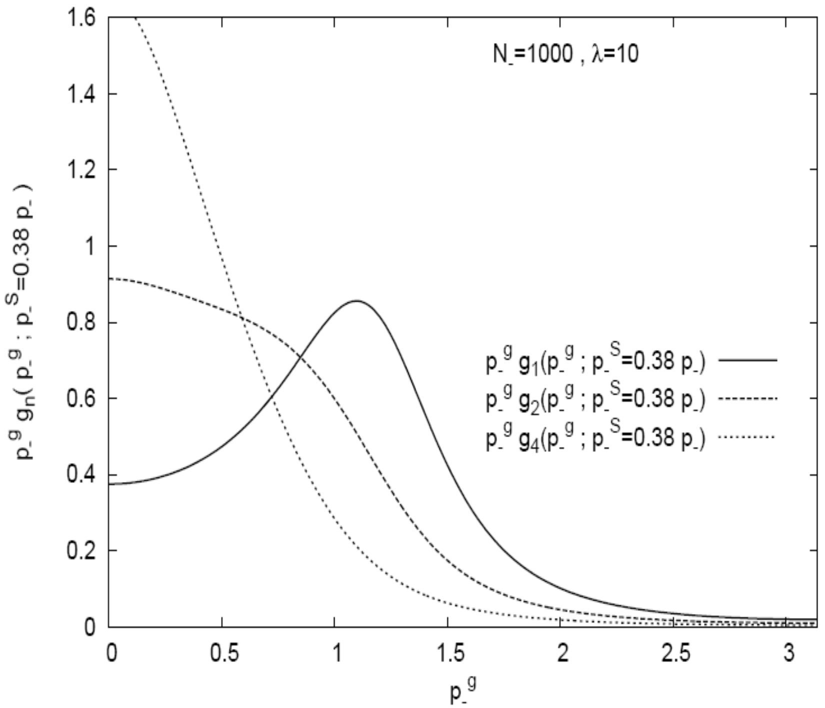
$$\begin{aligned} g_n(x_B) &= \int_{x_B}^1 \frac{dz_B}{z_B}\, g_{n-1}(x_B/z_B)\, P_{n\to n-1}(z_B) \\ x_B &= p_-^g/p_-^S \\ z_B &= p_-^{S\prime}/p_-^S \end{aligned}$$

Gluon distribution of a n-link dipole

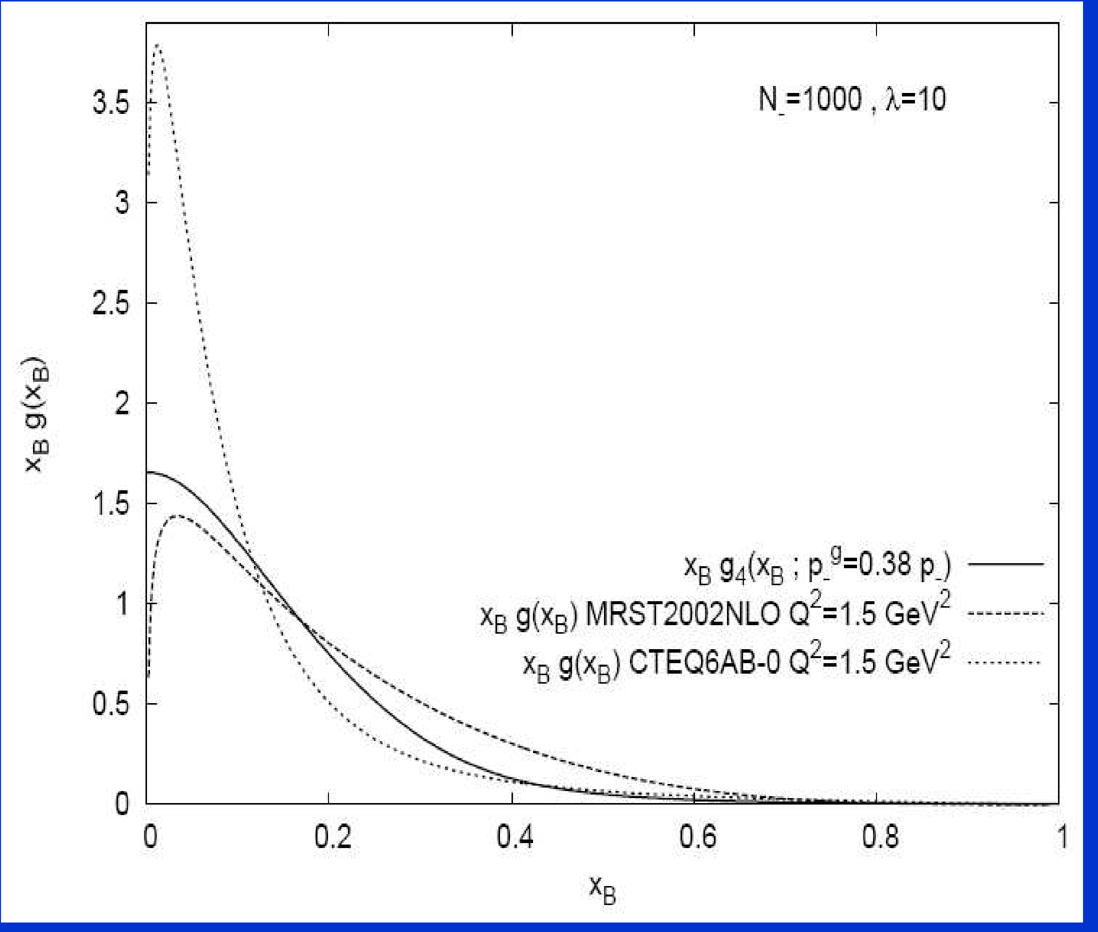
Pure phase space



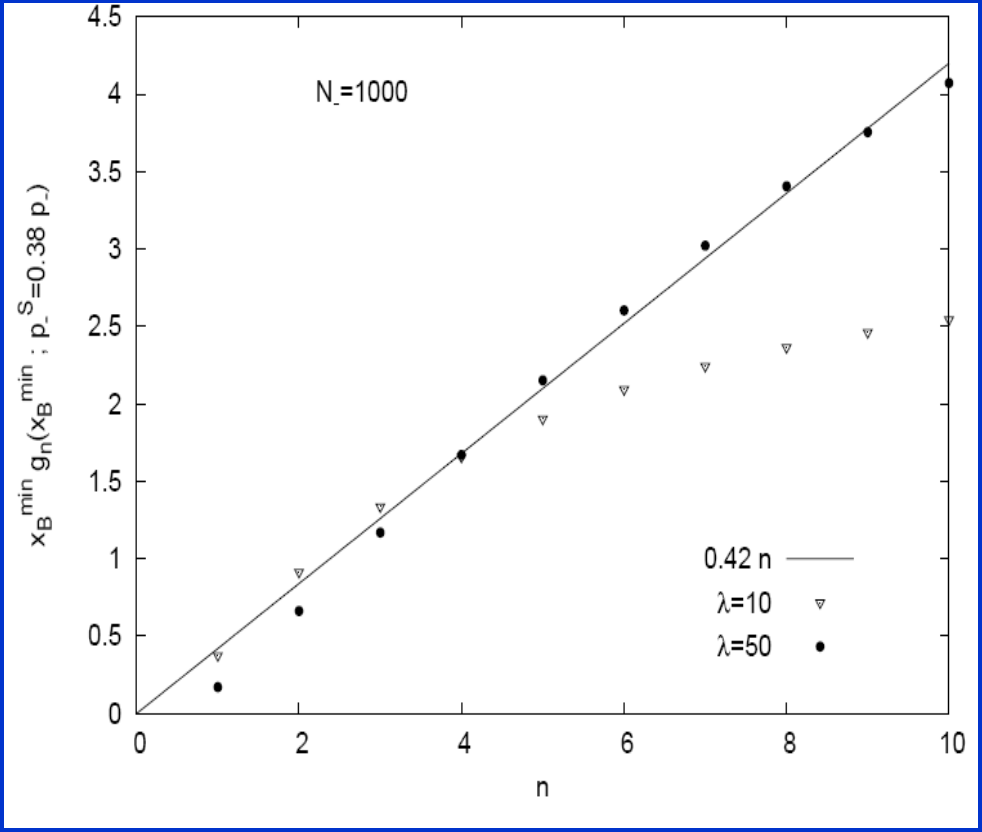
Full computation



Comparison with „experiment“



Low x behavior



Summary and conclusions:

- Near light cone coordinates are well suited to describe high energy scattering on the lattice. In particular, they allow in principle the determination of entire parton distribution functions
- Euclidean path integral treatments of the theory are not possible due to complex phases during the update process
- Ground state wave functionals have been constructed for strong and weak coupling which motivate a variational Ansatz valid over the whole coupling regime => A simplified ground state emerges
- We model a color dipole state equipped with longitudinal momentum on top of the variational ground state (non dynamical quarks)

- We find the full gluon distribution function g(xB) for this state
  - Large lattices are needed to observe scaling
  - It obeys a DGLAP type of evolution
  - Nice agreement with „experimental“ data
  - It is proportional to the size of the hadron at small x
- Outlook:
  - Use improved ground state wave functional in the gluonic sector

Thank you for your attention…

Backup Slides

Problems of LGT near the LC

- Euclidean gluonic Lagrange density

$$x^+ = -\mathrm{i}\, x_E^+ \qquad S = \mathrm{i}\int d^4x_E \mathcal{L}_E \equiv \mathrm{i} S_E \qquad Z = \int DA e^{-S_E}$$

$$\mathcal{L}_E \equiv \frac{1}{2} F^a_{+-} F^a_{+-} + \sum_k \left(\frac{\eta^2}{2} F^a_{+k} F^a_{+k} - \mathrm{i} F^a_{+k} F^a_{-k} \right) + \frac{1}{2} F^a_{12} F^a_{12}$$

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g\, f^{abc} A_\mu{}^b A_\nu{}^c$$

- a complex action remains (similar to finite baryonic density) -> sign problem
- Possible way out: Hamiltonian formulation

$\Longrightarrow$ Sampling of the ground state wavefunctional with guided diffusion quantum Monte-Carlo

$$\begin{aligned} |\Psi_0\rangle &= \lim_{t\to\infty} \exp\left[-t\left(\widehat{H}_0 - E\right)\right] |\Phi\rangle \\ &= \lim_{\substack{\Delta t\to 0 \\ N\Delta t\to\infty}} \prod_{n=1}^{N} \left\{ \exp\left[-\Delta t \left(\widehat{H}_0 - E\right)\right]\right\} |\Phi\rangle \end{aligned}$$

Analytic asymptotic solutions

- Strong coupling wavefunctional (perturbation theory)

$$\lambda = \frac{4}{g^4}$$

$$|\Psi_0\rangle = \prod_{\vec{x}} \exp\left\{\frac{1}{3}\,\lambda\,\tilde{\eta}^2 \mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right] \right.$$
$$\left. \frac{1}{16}\,\frac{\lambda}{1+\tilde{\eta}^2}\sum_k \Big(\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right]\Big)^2\right\}\left|\Psi_0^{(0)}\right\rangle + \mathcal{O}(\lambda^2)$$

  - Product state of single plaquette wavefunctionals

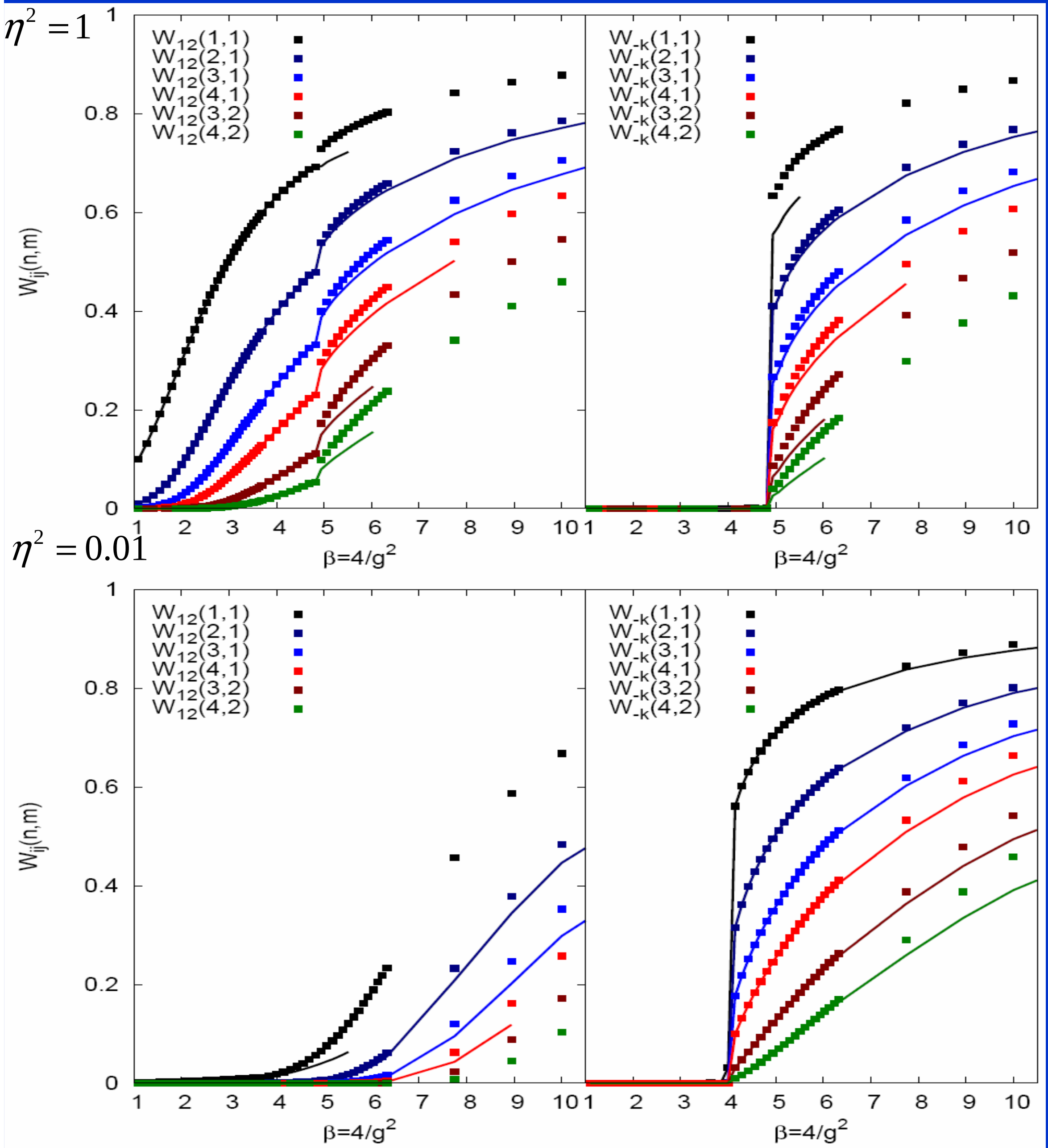
- Weak coupling wavefunctional

$$\Psi_0 = \exp\left\{-\sqrt{\lambda}\sum_{\vec{x},\vec{x}'}\sum_a \frac{1}{2}\vec{B}^a(\vec{x})\mathbf{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}')\frac{1}{2}\vec{B}^a(\vec{x}')\right\}$$

$$\mathbf{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}') \equiv \begin{pmatrix} \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 & 0 \\ 0 & \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 \\ 0 & 0 & \tilde{\eta}^2\gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') \end{pmatrix}$$

$$U_{ij}(\vec{x}) = \exp\left(\mathrm{i}F^a_{ij}(\vec{x})\lambda^a\right)$$
$$F^a_{ij}(\vec{x}) = \epsilon_{ijk}B^a_k(\vec{x}) + g\,f^{abc}A^b_i(\vec{x})A^c_j(\vec{x})$$
$$B^a_k(\vec{x}) = \epsilon_{klm}\left[A^a_m(\vec{x}) - A^a_m(\vec{x}-\vec{e}_l)\right]$$

  - Multivariate Gaussian wavefunctional



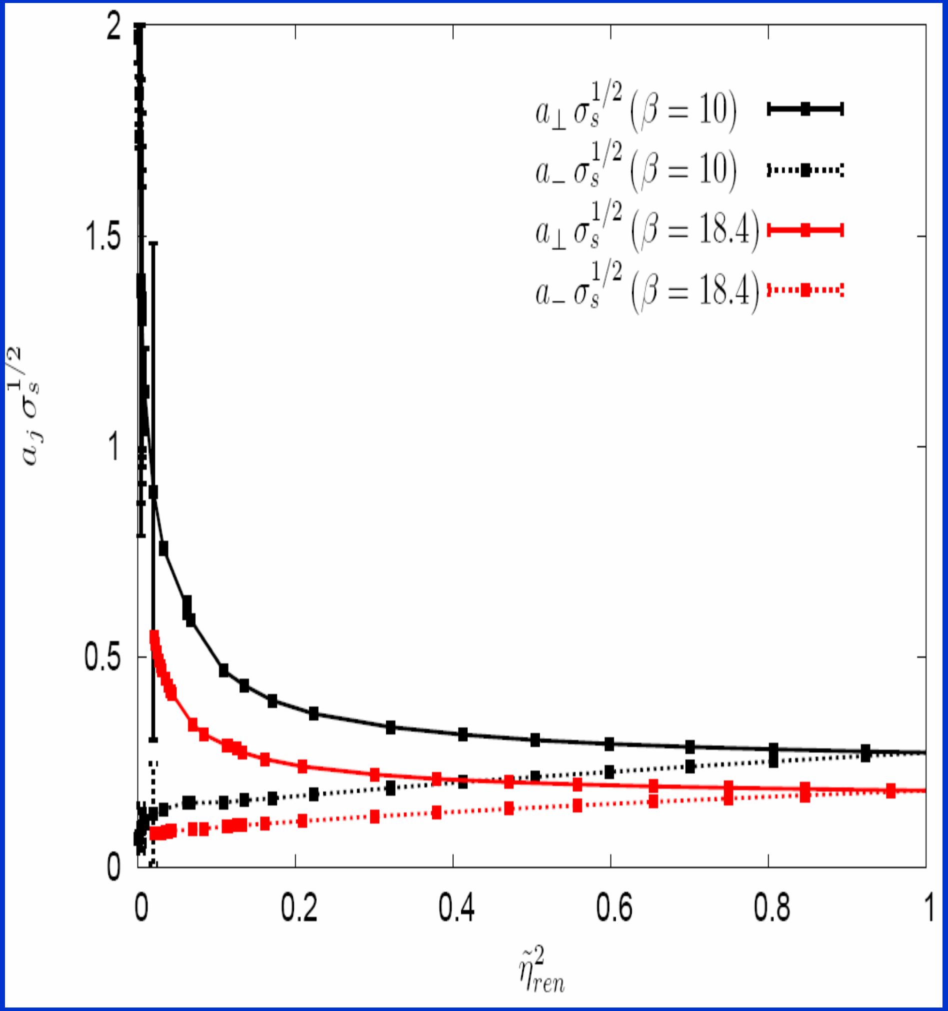
Wilson loop expectation values

$$\left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{-k}\right)\right]\right\rangle_{\Psi_0(\rho_0,\delta_0)} = \frac{I_2(4\,\rho_0)}{I_1(4\,\rho_0)}$$

$$\left\langle W_{ij}(n,m)\right\rangle_{\Psi_0(\rho_0,\delta_0)} = \left\langle \frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{ij}\right)\right]\right\rangle^{n\cdot m}_{\Psi_0(\rho_0,\delta_0)}$$

- Nice strong coupling behavior
- Better agreement to strong coupling for smaller values of η

Lattice spacings



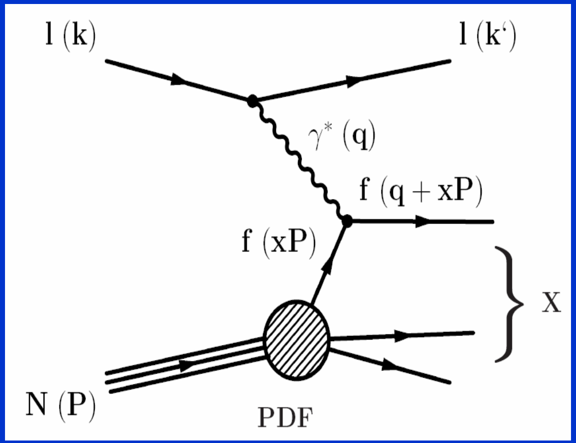
- $a_\perp = a_\perp(\beta, \eta) \quad \Rightarrow$
  the transversal lattice constant $a_\perp$ is varying with the boost parameter η

  $\Rightarrow$ UNPHYSICAL !

- Introduce two different couplings λ_- and $\lambda_\perp$ for the longitudinal and transversal part of the Hamiltonian
- $\Rightarrow$ three couplings $\lambda_-, \lambda_\perp, \eta$ which can be tuned in such a way that $a_\perp$ is independent of η_{ren}

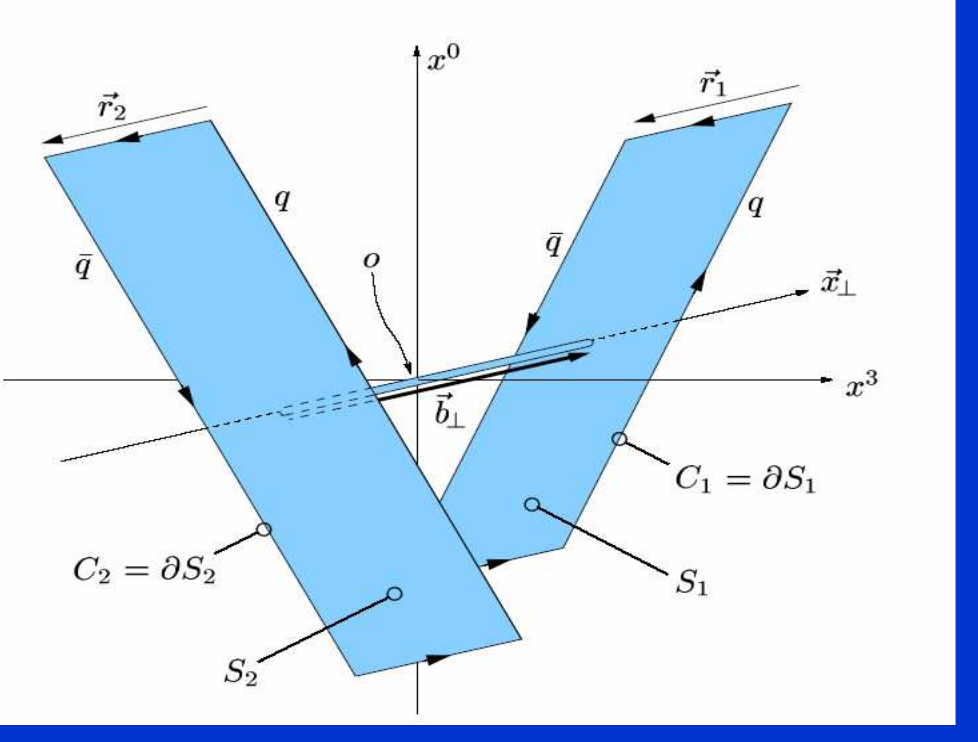
  a_- is $a_- = \eta_{ren}\, a_\perp$
- Work in progress

Hadron structure from DIS



Why Near Light Cone QCD ?

- Near light cone Wilson loop correlation functions determine the dipole-dipole cross section in QCD. By taking into account the hadronic wave functions one obtains hadronic cross sections



A. I. Shoshi, F. D. Steffen and H. J. Pirner, Nucl. Phys. A **709** (2002) 131.
O. Nachtmann, Annals Phys. **209** (1991) 436.

Motivation

- Near light-cone coordinates are a promising tool to investigate high energy scattering on the lattice
- NLC make high lab frame momenta accessible on the lattice with small a_-

$$P_- = -\eta P_3$$

$$P_j = \frac{2\pi}{N_j a_j} n_j \qquad n_j = 0, \ldots, N_j - 1$$

- Near light-cone QCD has a nontrivial vacuum which cannot be neglected even in the light-cone limit

Variational optimization

- Optimization procedure:
  - Find a first guess of the optimal parameters by an ordinary numerical minimization algorithm (Powell method)
  - Lay a lattice with 50 values in parameter space around this value and compute the correspondent energies
  - Close to the optimal value (minimum), the energy can be approximated by a quadratic form

$$\epsilon_0(a,b) \approx \epsilon_0(a_0,b_0) + \frac{1}{2}\begin{pmatrix} a-a_0 \\ b-b_0 \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} \frac{\partial^2\epsilon_0}{\partial_a\partial_a} & \frac{\partial^2\epsilon_0}{\partial_a\partial_b} \\ \frac{\partial^2\epsilon_0}{\partial_b\partial_a} & \frac{\partial^2\epsilon_0}{\partial_b\partial_b} \end{pmatrix} \cdot \begin{pmatrix} a-a_0 \\ b-b_0 \end{pmatrix}$$

  - Peform a fit to the quadratic form
  - Extract the optimal parameters and the energy

Continuum Hamiltonian and momentum

- Perform Legendre transformation of the Lagrange density for $A_+ = 0$:

$$\mathcal{L} = \sum_a \left[\frac{1}{2}F^a_{+-}F^a_{+-} + \sum_{k=1}^{2}\left(F^a_{+k}F^a_{-k} + \frac{\eta^2}{2}F^a_{+k}F^a_{+k}\right) - \frac{1}{2}F^a_{12}F^a_{12}\right]$$

$$\Pi^a_k = \frac{\delta\mathcal{L}}{\delta\partial_+ A^a_k} = \frac{\delta\mathcal{L}}{\delta F^a_{+k}} = F^a_{-k} + \eta^2 F^a_{+k}$$

$$\Pi^a_- = \frac{\delta\mathcal{L}}{\delta\partial_+ A^a_-} = \frac{\delta\mathcal{L}}{\delta F^a_{+-}} = F^a_{+-}$$

$$\left[\Pi^a_m(\vec{x}), A^b_n(\vec{y})\right] = -\mathrm{i}\delta^{ab}\delta_{mn}\delta^{(3)}(\vec{x}-\vec{y})$$

- Then, the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2}\sum_a \left[\Pi^a_-\Pi^a_- + F^a_{12}F^a_{12} + \sum_{k=1}^{2}\frac{1}{\eta^2}\left(\Pi^a_k - F^a_{-k}\right)^2\right]$$

- The Hamiltonian has to be supplemented by Gauss law

$$\left(D^{ab}_-\Pi^b_-(\vec{x}) + \sum_{k=1}^{2} D^{ab}_k\Pi^b_k(\vec{x})\right)|\Psi\rangle = 0 \quad \forall\, \vec{x}, a$$

$$E'_{\parallel}, B'_{\parallel} = E_{\parallel}, B_{\parallel}$$

$$E'_{\perp}, B'_{\perp} \propto \gamma E_{\perp}, \gamma B_{\perp}$$

$$\gamma = \frac{1}{\eta}$$

- Problem: Linear momentum operator term $\Pi^a_k F^a_{-k} + F^a_{-k}\Pi^a_k$ disturbs QDMC

- Solution: Momentum operator (obtained via the energy-momentum tensor)

$$\mathcal{P}_- = \frac{1}{2}\left(\Pi_k^a F_{-k}^a + F_{-k}^a \Pi_k^a\right)$$

- Commutation relations

$$[H, P_-] = 0$$
$$[H, G] = 0$$

  - Gauss law and P_- are constants of motion

- Choose trial state translation invariant -> $P_- |\Phi\rangle = 0$

$$|\Psi_0\rangle = \lim_{\tau\to\infty} \exp\left[-\left(\widehat{H} - E_0\right)\tau\right] |\Phi\rangle$$

-> Translation invariant ground state

- It is sufficient to consider H_{eff} for translation invariant trial states

$$\mathcal{H}_{eff} = \mathcal{H} + \frac{1}{\eta^2}\mathcal{P}_-$$
$$= \frac{1}{2}\sum_a \left[\Pi_-^a \Pi_-^a + F_{12}^a F_{12}^a + \sum_{k=1}^{2} \frac{1}{\eta^2}\left(\Pi_k^a \Pi_k^a + F_{-k}^a F_{-k}^a\right)\right]$$

$$|\Psi_0\rangle = \lim_{\tau\to\infty} \exp\left[-\left(H_{eff} - E_{eff}\right)\tau\right] |\Phi\rangle$$

- First explorative approach:
  Investigate H_{eff} variationally on the lattice

- Effective lattice Hamiltonian

$$\mathcal{H}_{\text{eff,lat}} = \frac{1}{N_- N_\perp^2}\frac{1}{a_\perp^4}\frac{2}{\sqrt{\lambda}}\sum_{\vec{x}}\left\{\frac{1}{2}\sum_a \Pi_-^a(\vec{x})^2 + \lambda\,\mathrm{Tr}\left[\mathbb{1} - \mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right.$$
$$\left. + \sum_{k,a}\frac{1}{2}\frac{1}{\tilde{\eta}^2}\left[\Pi_k^a(\vec{x})^2 + \lambda\,\mathrm{Tr}\left[\frac{\sigma_a}{2}\,\mathrm{Im}\Big(U_{-k}(\vec{x})\Big)\right]^2\right]\right\}$$

  - Additional global Z_2 invariance in comparison to the full lattice Hamiltonian

$$U_k(\vec{x}_\perp, x^-) \longrightarrow z\,U_k(\vec{x}_\perp, x^-) \quad \forall\ \vec{x}_\perp \text{ and } x^- \text{ fixed},\quad z \in Z_2$$
$$U_{-k}(\vec{x}_\perp, x^-) \longrightarrow z\,U_{-k}(\vec{x}_\perp, x^-)$$

  - Restrict to the spontaneously broken phase

$$\left\langle \mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}\Big)\right]\right\rangle \begin{cases} = 0 & Z_2 \text{ symmetric phase} \\ \neq 0 & Z_2 \text{ broken phase}\end{cases}$$

← order parameter

- Lattice longitudinal momentum operator

$$\mathcal{P}_- \equiv \frac{1}{N_- N_\perp^2}\frac{1}{\xi^2}\frac{1}{a_\perp^4}\sum_{\vec{x},k,a}\left(\Pi_k^a(\vec{x})\cdot\mathrm{Tr}\left[\frac{\sigma_a}{2}\,\mathrm{Im}\Big(U_{-k}(\vec{x})\Big)\right]\right.$$
$$\left. + \mathrm{Tr}\left[\frac{\sigma_a}{2}\,\mathrm{Im}\Big(U_{-k}(\vec{x})\Big)\right]\cdot\Pi_k^a(\vec{x})\right)$$

  - Not the generator of lattice translations -> check substitution $H_{lat} \rightarrow H_{eff,lat}$

$$\left[P'_{-,\text{lat}}, H_{\text{lat}}\right] = 0 + \mathcal{O}(a_-^2)$$

Strong Coupling Solution

- Potential energy <-> small perturbation

$$\mathcal{H}_{\text{eff,lat}} = \mathcal{T} + \lambda\,\mathcal{V}_{\text{pot}}$$

$$\mathcal{T} = \frac{1}{N_- N_\perp^2}\frac{1}{a_\perp^4}\frac{2}{\sqrt{\lambda}}\sum_{\vec{x},a}\left[\frac{1}{2}\frac{1}{\tilde{\eta}^2}\sum_k \Pi_k^a(\vec{x})^2 + \frac{1}{2}\,\Pi_-^a(\vec{x})^2\right]$$

$$\mathcal{V}_{\text{pot}} = \frac{1}{N_- N_\perp^2}\frac{1}{a_\perp^4}\frac{2}{\sqrt{\lambda}}\sum_{\vec{x}}\left\{\frac{1}{2}\frac{1}{\tilde{\eta}^2}\sum_k\left[1-\frac{1}{4}\Big(\mathrm{Tr}\left[\mathrm{Re}\big(U_{-k}(\vec{x})\big)\right]\Big)^2\right] + \left[1-\frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\big(U_{12}(\vec{x})\big)\right]\right]\right\}$$

- Ground state of $\mathcal{T}$

$$\Pi_j^a(\vec{x})\left|\Psi_0^{(0)}\right\rangle = 0 \quad \forall\,\vec{x},a \;\wedge\; \forall\, j\in\{1,2,-\} \qquad E_0^{(0)} = 0$$

- Ordinary Rayleigh-Schrödinger perturbation theory:

$$\left|\Psi_0^{(1)}\right\rangle = \frac{1}{E_0^{(0)} - T}V_{\text{pot}}\left|\Psi_0^{(0)}\right\rangle \qquad E_0^{(1)} = \left\langle\Psi_0^{(0)}\right|V_{\text{pot}}\left|\Psi_0^{(0)}\right\rangle$$

Strong Coupling Solution

- $$\mathcal{H}_{\mathrm{eff,lat}} = \mathcal{T} + \lambda\,\mathcal{V}_{\mathrm{pot}}$$

- Strong coupling wavefunctional

$$|\Psi_0\rangle = \prod_{\vec{x}} \exp\left\{\frac{1}{3}\,\lambda\,\tilde{\eta}^2 \mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right. \\ \left.\frac{1}{16}\,\frac{\lambda}{1+\tilde{\eta}^2}\sum_k \Big(\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right]\Big)^2\right\}\left|\Psi_0^{(0)}\right\rangle + \mathcal{O}(\lambda^2)$$

  - Product state of single plaquette excitations
  - LC limit: transversal dynamics decouple

    -> Effective reduction to a 2-dim (spatial) theory

- Energy density in the strong coupling limit:

$$\epsilon_0 = \frac{2}{a_\perp^4\,\tilde{\eta}^2}\left(\frac{3}{4}+\tilde{\eta}^2\right)\sqrt{\lambda} + \mathcal{O}(\lambda^{3/2})$$

Weak Coupling Solution

- Effective lattice Hamiltonian

$$\mathcal{H}_{\text{eff,lat}} = \frac{1}{N_- N_\perp^2} \frac{1}{a_\perp^4} \frac{2}{\sqrt{\lambda}} \sum_{\vec{x}} \left\{ \frac{1}{2} \sum_a \Pi_-^a(\vec{x})^2 + \lambda \operatorname{Tr}\left[1\!\!1 - \operatorname{Re}\Big(U_{12}(\vec{x}) \Big) \right] + \sum_{k,a} \frac{1}{2} \frac{1}{\tilde{\eta}^2} \left[\Pi_k^a(\vec{x})^2 + \lambda \operatorname{Tr}\left[\frac{\sigma_a}{2} \operatorname{Im}\Big(U_{-k}(\vec{x}) \Big) \right]^2 \right] \right\} . \quad (7$$

$$U_{ij}(\vec{x}) = \exp\left(\mathrm{i} F_{ij}^a(\vec{x}) \lambda^a \right)$$

$$F_{ij}^a(\vec{x}) = \epsilon_{ijk} B_k^a(\vec{x}) + g\, f^{abc} A_i^b(\vec{x}) A_j^c(\vec{x})$$

$$B_k^a(\vec{x}) = \epsilon_{klm} \left[A_m^a(\vec{x}) - A_m^a(\vec{x} - \vec{e}_l) \right]$$

- Rescale the fields and momenta -> Expansion

$$\widetilde{A}_i^a(\vec{x}) = \sqrt{\lambda}\, \widehat{A}_i^a(\vec{x}) \quad \Rightarrow \quad \widetilde{B}_i^a(\vec{x}) = \sqrt{\lambda}\, \widehat{B}_i^a(\vec{x})$$

$$\widetilde{\Pi}_i^a(\vec{x}) = \frac{1}{\sqrt{\lambda}} \widehat{\Pi}_i^a(\vec{x})$$

$$\mathcal{H}_{\text{eff,lat}} = \frac{1}{a_\perp^4} \frac{1}{\sqrt{\lambda}} \sum_{\vec{x},a} \frac{1}{N_- N_\perp^2} \left\{ \lambda\, \vec{E}^a(\vec{x})^\dagger \cdot \vec{E}^a(\vec{x}) + \frac{1}{4} \vec{Q}^a(\vec{x})^\dagger \cdot \vec{Q}^a(\vec{x}) \right\} + \mathcal{O}\left(\frac{1}{\lambda^{5/4}} \right)$$

Weak Coupling Solution

- $$U_{ij}(\vec{x}) = \exp\left(\mathrm{i}F^a_{ij}(\vec{x})\lambda^a\right)$$
  $$F^a_{ij}(\vec{x}) = \epsilon_{ijk}B^a_k(\vec{x}) + g\, f^{abc}A^b_i(\vec{x})A^c_j(\vec{x})$$
  $$B^a_k(\vec{x}) = \epsilon_{klm}\left[A^a_m(\vec{x}) - A^a_m(\vec{x}-\vec{e}_l)\right]$$

- Weak coupling wavefunctional

$$\Psi_0 = \exp\left\{-\sqrt{\lambda}\sum_{\vec{x},\vec{x}'}\sum_a \frac{1}{2}\vec{B}^a(\vec{x})\mathbf{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}')\frac{1}{2}\vec{B}^a(\vec{x}')\right\}$$

$$\mathbf{\Gamma}_{\tilde{\eta}}(\vec{x}-\vec{x}') \equiv \begin{pmatrix} \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 & 0 \\ 0 & \gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') & 0 \\ 0 & 0 & \tilde{\eta}^2\gamma_{\tilde{\eta}}(\vec{x}-\vec{x}') \end{pmatrix}$$

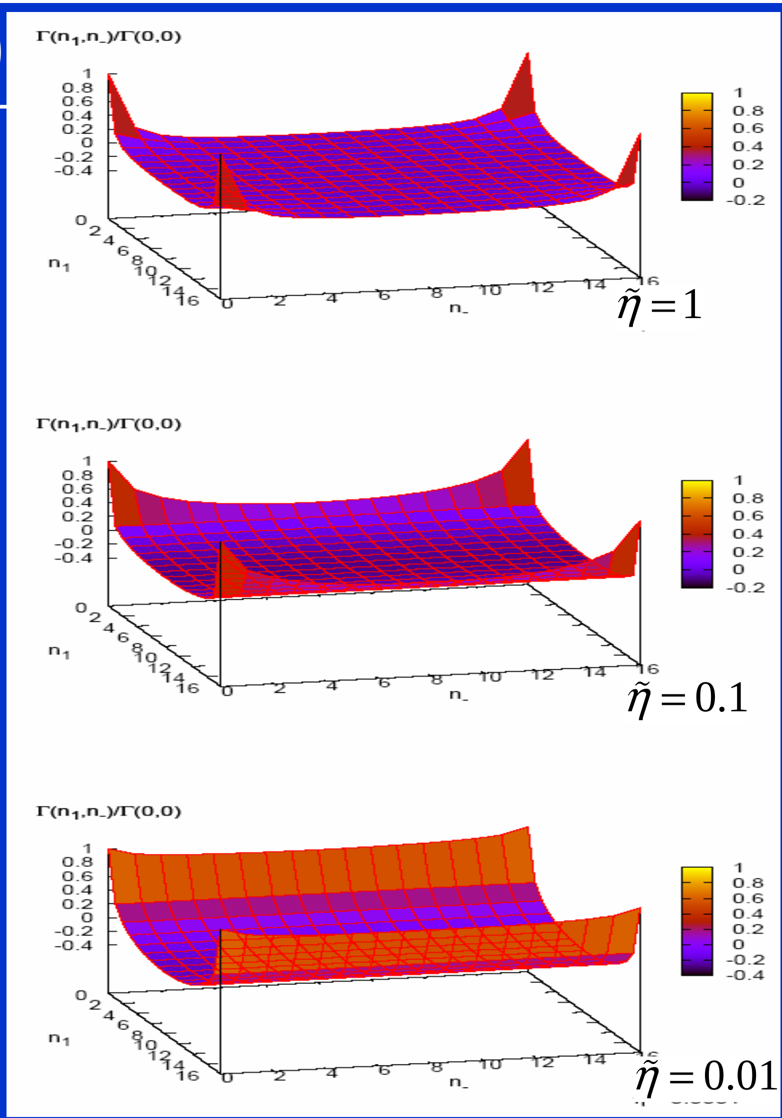
  - Multivariate Gaussian wavefunctional
  - LC limit: transversal dynamics decouple

    -> Effective reduction to a 2-dim (spatial) theory

- Energy density in the weak coupling limit:

$$\epsilon_0 = \frac{1}{a_\perp^4}\frac{6}{\tilde{\eta}^2}\sum_{\vec{k}}\frac{1}{N_- N_\perp^2}\left(\tilde{\eta}^2 s_1^2 + \tilde{\eta}^2 s_2^2 + s_3^2\right)^{1/2}$$

$$\frac{\gamma_{\tilde{\eta}}(\Delta \vec{x})}{\gamma_{\tilde{\eta}}(\vec{0})}$$

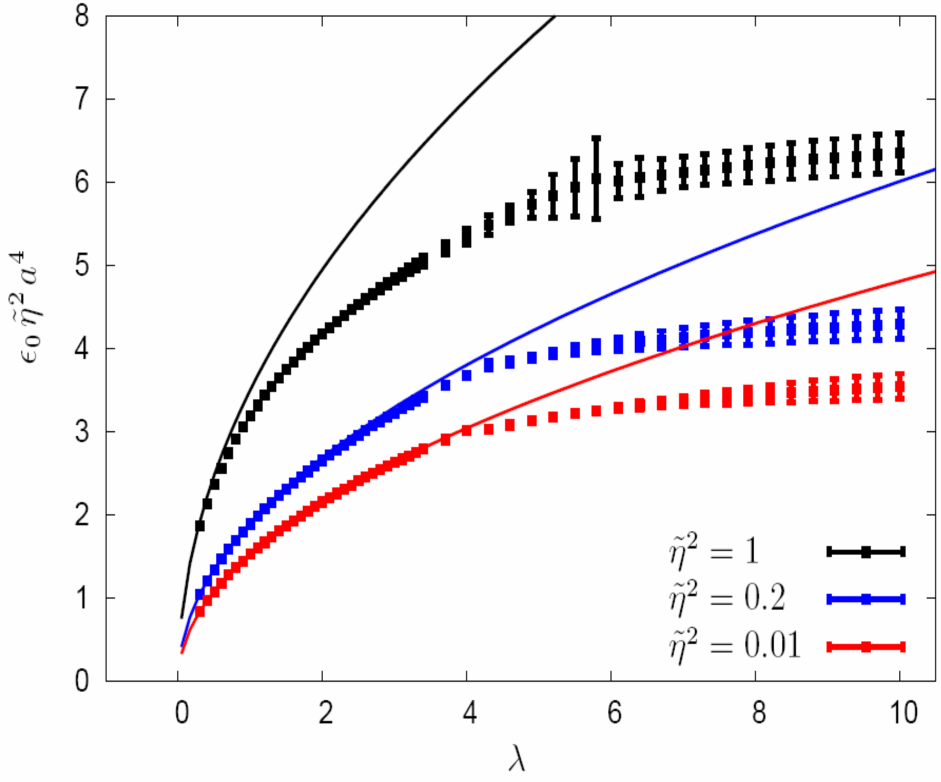


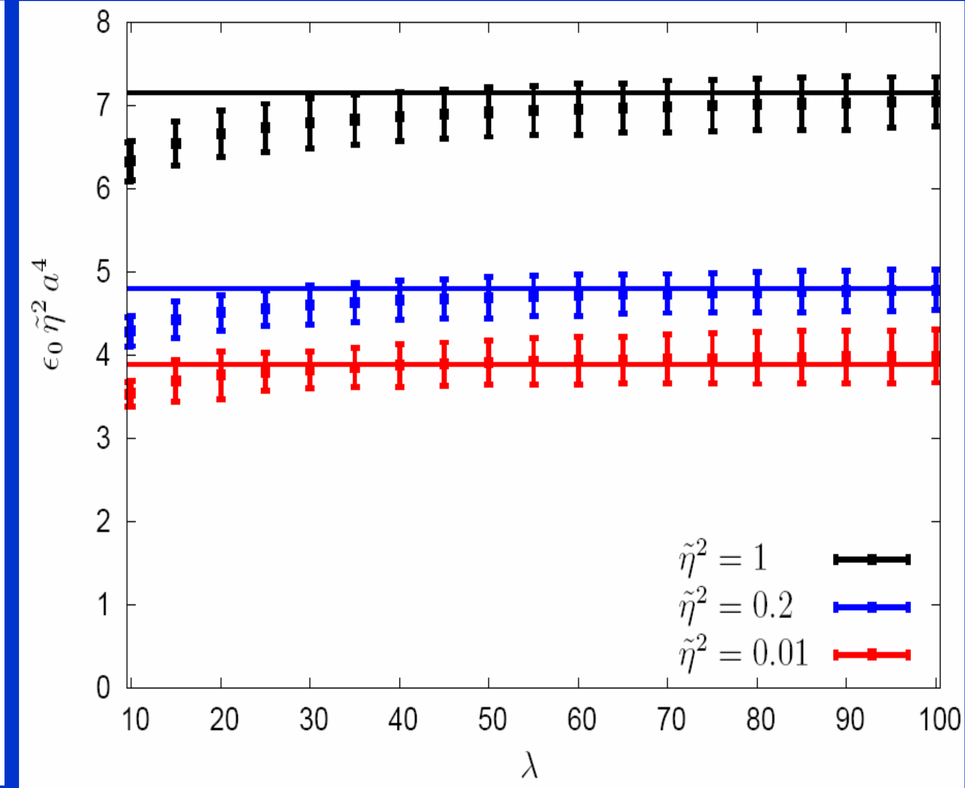
- Equal time theories:
  Covariance matrix weakly off-diagonal
  - Product of single site wavefunctionals suitable
- Decreasing $\tilde{\eta}$
  - Correlations among longitudinally separated plaquettes become increasingly important
- LC Limit
  - Each plaquette is equally correlated with every other longitudinally separated plaquette

- Trial wavefunctional (Restrict to product of single site wavefunctionals)

$$\Psi_0(\rho,\delta) = \prod_{\vec{x}} \exp\left\{ \sum_{k=1}^{2} \rho \,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right] + \delta\,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right\}$$

Lattice Energy density 16x16x16





- Nice agreement between prediction/measurement for all values of $\tilde{\eta}$

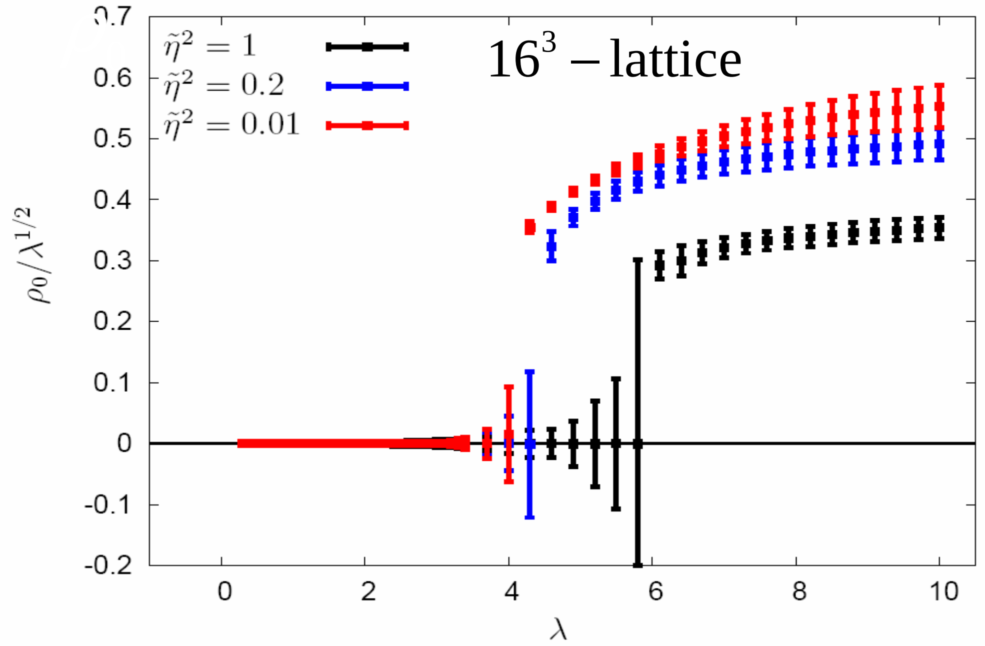
Strong coupling prediction (solid line):

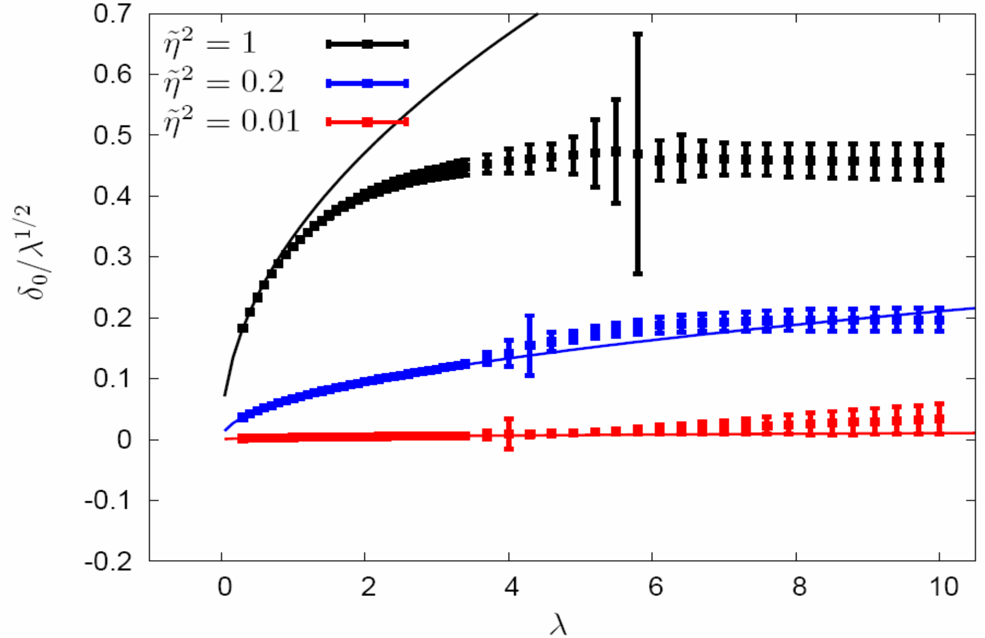
$$\epsilon_0 = \frac{2}{a_\perp^4\,\tilde{\eta}^2}\left(\frac{3}{4}+\tilde{\eta}^2\right)\sqrt{\lambda}+\mathcal{O}(\lambda^{3/2})$$

Weak coupling prediction (solid line):

$$\epsilon_0 = \frac{1}{a_\perp^4}\frac{6}{\tilde{\eta}^2}\sum_{\vec{k}}\frac{1}{N_-N_\perp^2}\left(\tilde{\eta}^2 s_1^2+\tilde{\eta}^2 s_2^2+s_3^2\right)^{1/2}$$

Optimal ρ_0 and δ_0 (strong coupling)





$$\Psi_0(\rho,\delta) = \prod_{\vec{x}} \exp\left\{\sum_{k=1}^{2} \rho \,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}(\vec{x})\Big)\right] + \delta\,\mathrm{Tr}\left[\mathrm{Re}\Big(U_{12}(\vec{x})\Big)\right]\right\}$$

- Strong coupling limit is nicely reproduced (solid line)
- $\rho_0 = 0$
- Remember the order parameter

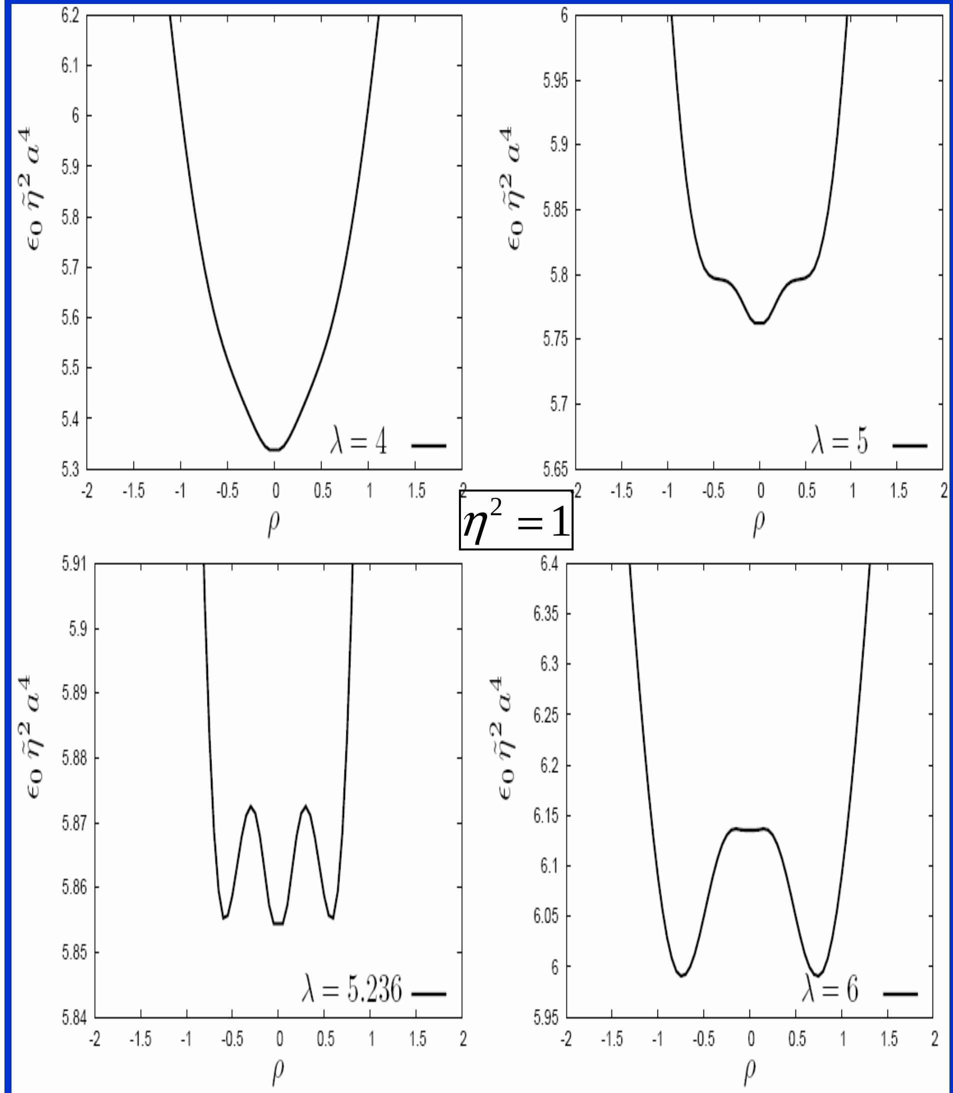
$$\left\langle \mathrm{Tr}\left[\mathrm{Re}\Big(U_{-k}\Big)\right]\right\rangle \begin{cases} = 0 & Z_2 \text{ symmetric phase} \\ \neq 0 & Z_2 \text{ broken phase} \end{cases}$$

for the associated phases of the discussed Z_2 transformation

$$U_k(\vec{x}_\perp, x^-) \to z\, U_k(\vec{x}_\perp, x^-)$$

- How does the critical coupling behave with the light-cone distance (more qantitatively)

Phase transition



- Energy density for fixed optimal δ_0 as a function of ρ for different values of λ
- Z_2 trafo corresponds to $\rho \to -\rho$
- single minimum turns into two degenerate minima which differ by a Z_2 trafo
- 1st order phase transition in accordance with the Ehrenfest classification
- Analytic estimate (strong coupling)

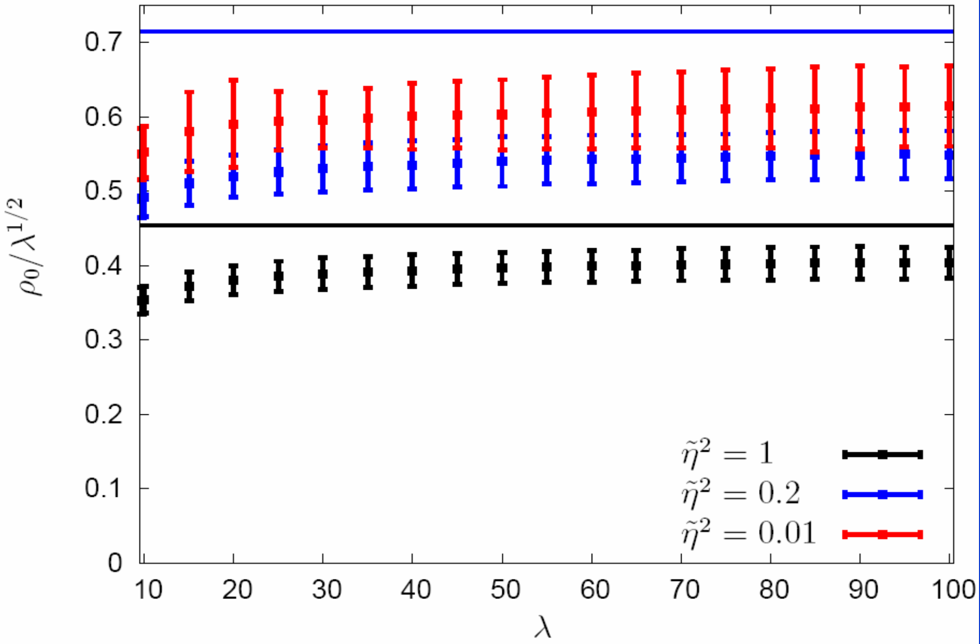
$$\left\langle \left(\frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{-k}\right)\right]\right)\right\rangle_{\Psi_0(\rho,\delta)} \approx \rho\left(1 - \frac{2}{3}\rho^2 + \frac{2}{3}\rho^4\right)$$

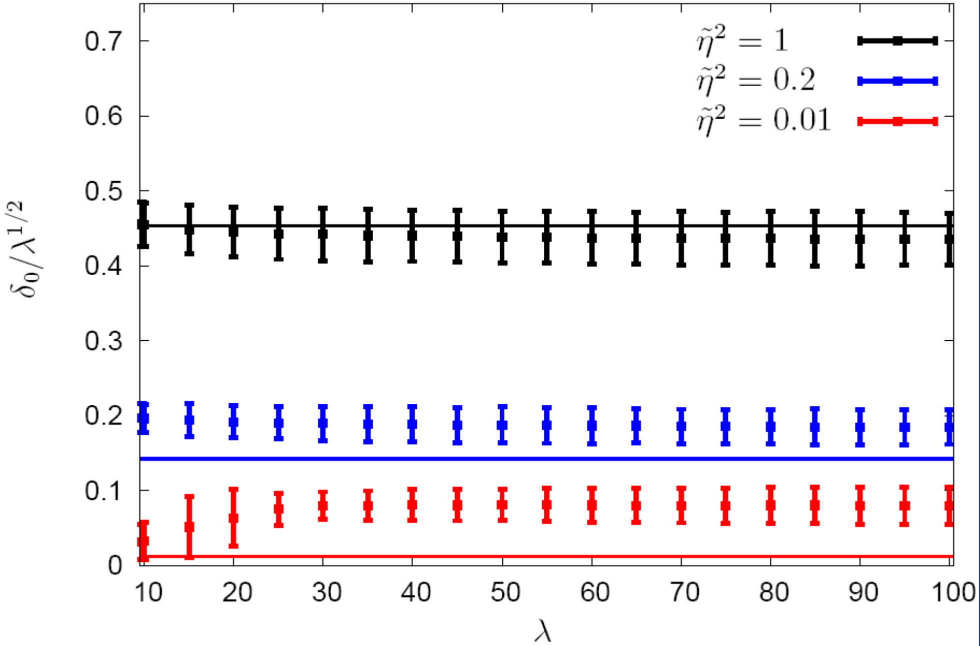
$$\left\langle \left(\frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(U_{-k}\right)\right]\right)^2\right\rangle_{\Psi_0(\rho,\delta)} \approx \frac{1}{4} + \frac{1}{2}\rho^2 - \frac{1}{2}\rho^4 + \frac{8}{15}\rho^6$$

$$\Rightarrow \quad \lambda_c(\tilde{\eta}^2) \approx 3\left(1 + \tilde{\eta}^2\right)$$

- By choosing $\lambda > \lambda_c$ fixed we are able to decrease η without crossing the critical line

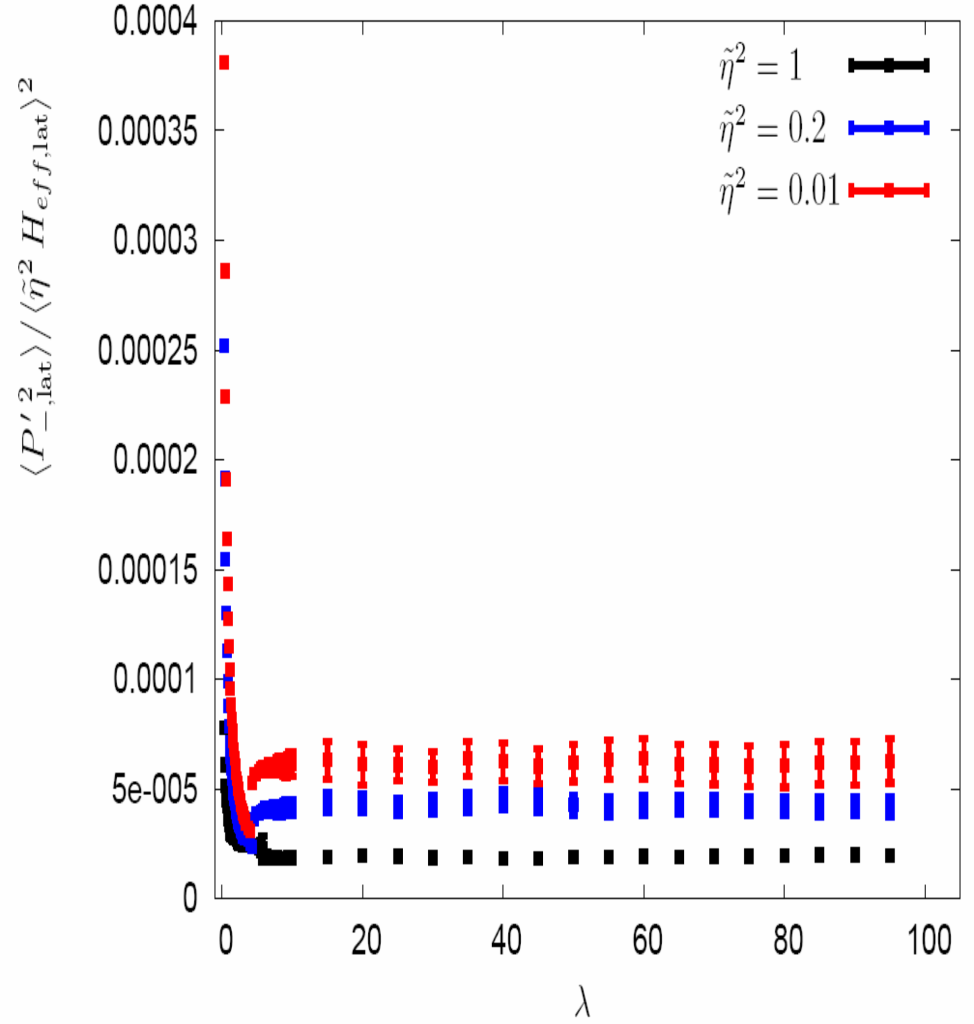
Optimal ρ_0 and δ_0 (weak coupling)





- Assymptotic weak coupling behavior seen for $\tilde{\eta} = 1$
- Increasing disagreement in the weak coupling regime for decreasing $\tilde{\eta}$
    - Only effective description possible

P_- expectation values



- How close is P_- to the exact generator of lattice translations ? (important for the applicability of QDMC)
- For every purely real valued wave-functional ψ_0 we have

$$\langle \Psi_0 | \, P'_{-,\mathrm{lat}} \, | \Psi_0 \rangle \;=\; 0$$

which follows from partial integration and is consistent with an exact eigen-state

$$\langle \Psi_0 | \, \Pi^a_j(\vec{y}) \, g(U) \, | \Psi_0 \rangle \;=\; - \langle \Psi_0 | \, g(U) \, \Pi^a_j(\vec{y}) \, | \Psi_0 \rangle$$

- Look at the second moment $\langle \Psi_0 | \, P'^{\,2}_{-,\mathrm{lat}} \, | \Psi_0 \rangle$
- Fluctuations of P_- are always less than 1% of the total energy around its mean value equal to zero and may be neglected in realistic computations

$$\Psi_0 \propto \exp\left\{\sqrt{\lambda}\sum_{\vec{x},\vec{x}'}\sum_{k}\Gamma^{kk}_{\tilde{\eta}}(\vec{x}-\vec{x}')R_k(\vec{x},\vec{x}')\right\}$$

$$R_k(\vec{x},\vec{x}') = \frac{1}{2}|\epsilon_{kij}| \cdot \begin{cases} \mathrm{Tr}\left[\mathrm{Re}\left(\cdots \right)\right] & \text{for } \vec{x}=\vec{x}' \text{ and} \\ \frac{1}{\#p}\sum_{\forall p}\frac{1}{2}\mathrm{Tr}\left[\mathrm{Re}\left(\cdots - \cdots - \cdots \right)\right] & \text{for } \vec{x}\neq\vec{x}' \end{cases}$$