D Umformungen zur ART

D.a Von (41.4) zu (41.12)

Wir leiten die Darstellung des Krümmungstensors (41.12) aus der Darstellung (41.4) her.

$$\begin{aligned}
R_{\lambda\mu\nu\kappa} &= g_{\lambda\iota}R^{\iota}{}_{\mu\nu\kappa} \\
&= g_{\lambda\iota}\partial_\nu\Gamma^{\iota}{}_{\mu\kappa} - g_{\lambda\iota}\partial_\kappa\Gamma^{\iota}{}_{\mu\nu} + g_{\lambda\iota}(\Gamma^{\rho}{}_{\mu\kappa}\Gamma^{\iota}{}_{\rho\nu} - \Gamma^{\rho}{}_{\mu\nu}\Gamma^{\iota}{}_{\rho\kappa}) \\
&= \partial_\nu(g_{\lambda\iota}\Gamma^{\iota}{}_{\mu\kappa}) - \partial_\kappa(g_{\lambda\iota}\Gamma^{\iota}{}_{\mu\nu}) - (\partial_\nu g_{\lambda\iota})\Gamma^{\iota}{}_{\mu\kappa} + (\partial_\kappa g_{\lambda\iota})\Gamma^{\iota}{}_{\mu\nu} + \Gamma^{\rho}{}_{\mu\kappa}\Gamma_{\lambda\rho\nu} - \Gamma^{\rho}{}_{\mu\nu}\Gamma_{\lambda\rho\kappa} \\
&= \partial_\nu\Gamma_{\lambda\mu\kappa} - \partial_\kappa\Gamma_{\lambda\mu\nu} \\
&\quad \underbrace{-\Gamma_{\lambda\iota\nu}\Gamma^{\iota}{}_{\mu\kappa}}_{*} - \Gamma_{\iota\lambda\nu}\Gamma^{\iota}{}_{\mu\kappa} + \underbrace{\Gamma_{\lambda\iota\kappa}\Gamma^{\iota}{}_{\mu\nu}}_{**} + \Gamma_{\iota\kappa\lambda}\Gamma^{\iota}{}_{\mu\nu} + \underbrace{\Gamma^{\rho}{}_{\mu\kappa}\Gamma_{\lambda\rho\nu}}_{*} - \underbrace{\Gamma^{\rho}{}_{\mu\nu}\Gamma_{\lambda\rho\kappa}}_{**} \\
&= \frac{1}{2}(\partial_\nu\partial_\mu g_{\lambda\kappa} + \underbrace{\partial_\nu\partial_\kappa g_{\lambda\mu}}_{***} - \partial_\nu\partial_\lambda g_{\mu\kappa} - \partial_\kappa\partial_\mu g_{\nu\lambda} - \underbrace{\partial_\kappa\partial_\nu g_{\lambda\mu}}_{***} + \partial_\kappa\partial_\lambda g_{\mu\nu}) \\
&\quad -\Gamma_{\iota\lambda\nu}\Gamma^{\iota}{}_{\mu\kappa} + \Gamma_{\iota\kappa\lambda}\Gamma^{\iota}{}_{\mu\nu}
\end{aligned} \tag{D.1}$$

Es heben sich dabei jeweils die Terme mit gleicher Anzahl von Sternen weg. Dadurch verschwinden die ursprünglichen ΓΓ-Beiträge und neue entstehen.

D.b Herleitung von (45.8)

Wir leiten hier explizit die Ableitung (45.8) her. Unter Berücksichtigung, dass wir bezüglich der Indizes μ und ν symmetrisieren müssen, erhalten wir

$$\frac{\partial g_{\lambda\pi}}{\partial g^{\mu\nu}} = -\frac{1}{2}(g_{\lambda\mu}g_{\nu\pi} + g_{\lambda\nu}g_{\mu\pi}), \tag{D.2}$$

$$\frac{\partial g_{\rho\iota}}{\partial g^{\mu\nu}} = -\frac{1}{2}(g_{\rho\mu}g_{\iota\nu} + g_{\rho\nu}g_{\iota\mu}), \tag{D.3}$$

$$\frac{\partial g^{\sigma\tau}}{\partial g^{\mu\nu}} = \frac{1}{2}(\delta^\sigma_\mu\delta^\tau_\nu + \delta^\sigma_\nu\delta^\tau_\mu). \tag{D.4}$$

Damit folgt

$$\begin{aligned}
\frac{\partial\Gamma^{\sigma}{}_{\rho\lambda}}{\partial g^{\mu\nu}} &= \frac{1}{2}(g_{\lambda\mu}g_{\pi\nu} + g_{\lambda\nu}g_{\pi\mu})(\partial_\rho g^{\sigma\pi} - g^{\sigma\tau}g_{\rho\iota}\partial_\tau g^{\iota\pi}) \\
&+ \frac{1}{2}(g_{\rho\mu}g_{\iota\nu} + g_{\rho\nu}g_{\iota\mu})(\partial_\lambda g^{\iota\sigma} - g^{\sigma\tau}g_{\lambda\pi}\partial_\tau g^{\iota\pi}) \\
&+ \frac{1}{4}(\delta^\sigma_\mu\delta^\tau_\nu + \delta^\sigma_\nu\delta^\tau_\mu)g_{\rho\iota}g_{\lambda\pi}\partial_\tau g^{\iota\pi}.
\end{aligned} \tag{D.5}$$

Mit

$$\partial_\tau g^{\iota\pi} = -g^{\iota\kappa}g^{\pi\kappa'}\partial_\tau g_{\kappa\kappa'} = -g^{\iota\kappa}g^{\pi\kappa'}(\Gamma_{\kappa\kappa'\tau} + \Gamma_{\kappa'\kappa\tau}) = -g^{\pi\kappa}\Gamma^{\iota}{}_{\kappa\tau} - g^{\iota\kappa}\Gamma^{\pi}{}_{\kappa\tau} \tag{D.6}$$

D Transformations to GTR

D.a From (41.4) to (41.12)

We derive the representation (41.12) of the curvature tensor from the representation (41.4).

The terms with equal number of stars cancel. In this way the original ΓΓ contributions disappear and new ones show up.

D.b Derivation of (45.8)

Here we derive explicitly (45.8). Considering that we have to symmetrize with respect to μ and ν we obtain

This yields

With

folgt dann

one derives

$$\begin{aligned}\frac{\partial\Gamma^{\sigma}{}_{\rho\lambda}}{\partial g^{\mu\nu}} &= -\frac{1}{4}(g_{\lambda\mu}g_{\nu\pi}+g_{\lambda\nu}g_{\mu\pi})(g^{\pi\kappa}\Gamma^{\sigma}{}_{\rho\kappa}+\underbrace{g^{\sigma\kappa}\Gamma^{\pi}{}_{\rho\kappa}-g^{\sigma\tau}g_{\rho\iota}g^{\iota\kappa}\Gamma^{\pi}{}_{\kappa\tau}}_{0}-g^{\sigma\tau}g_{\rho\iota}g^{\pi\kappa}\Gamma^{\iota}{}_{\kappa\tau})\\ &\quad -\frac{1}{4}(g_{\rho\mu}g_{\iota\nu}+g_{\rho\nu}g_{\iota\mu})(g^{\iota\kappa}\Gamma^{\sigma}{}_{\kappa\lambda}+\underbrace{g^{\sigma\kappa}\Gamma^{\iota}{}_{\kappa\lambda}-g^{\sigma\tau}g_{\lambda\pi}g^{\pi\kappa}\Gamma^{\iota}{}_{\kappa\tau}}_{0}-g^{\sigma\tau}g_{\lambda\pi}g^{\iota\kappa}\Gamma^{\pi}{}_{\kappa\tau})\\ &\quad -\frac{1}{4}(\delta^{\sigma}_{\mu}\delta^{\tau}_{\nu}+\delta^{\sigma}_{\nu}\delta^{\tau}_{\mu})(g_{\rho\iota}g_{\lambda\pi}g^{\pi\kappa}\Gamma^{\iota}{}_{\kappa\tau}+g_{\rho\iota}g_{\lambda\pi}g^{\iota\kappa}\Gamma^{\pi}{}_{\kappa\tau}).\end{aligned} \tag{D.7}$$

Die Terme mit den δ-Funktionen ergeben die erste Zeile von (45.8), die Terme mit dem Faktor $g^{\sigma\tau}$ die dritte Zeile und die übrigen die zweite Zeile.

The terms with the δ functions yield the first line of (45.8), the terms with the factor $g^{\sigma\tau}$ yield the third line, and the other terms the second line.