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An exactly solvable model for equilibration in bosonic systems

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Abstract – An analytical model for the kinetic equilibration of finite Bose systems is outlined. The corresponding transport equation is solved exactly through a nonlinear transformation. The model is applied to the equilibration of a cold quantum gas including implicitly the formation of a Bose-Einstein condensate through particle-number conservation.

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Introduction. – Nonequilibrium processes in finite Bose systems are of great current interest. Due to the quantum-statistical properties of bosons, condensation can accompany thermalization even in case of weak interactions, when the final state deviates from a purely thermal distribution significantly. The nonlinear kinetic stage has been analyzed, in particular, in refs. [1,2]. Eventually all particles populate the condensed state [3,4] when the temperature T approaches zero and the thermal cloud vanishes.

The evolution of the distribution function towards the one in the presence of a condensate and the time scale for its appearance has also been considered in refs. [5,6], and the kinetics of Bose-Einstein condensation in a trap was discussed in, *e.g.*, refs. [7,8]. A thorough theoretical review is found in ref. [9], a nonlinear Schrödinger model [10,11] in ref. [12]. The buildup of coherence in addition to kinetic equilibration was described in ref. [13]. A review of some of the kinetic theories is in ref. [14].

This letter presents a nonlinear model for the kinetic evolution of a Bose gas that is exactly solvable in the limit of constant transport coefficients, and thus can replace the relaxation ansatz that does not properly account for the inherent nonlinearity of the problem. Condensate formation is not treated explicitly, but only through particle-number conservation.

The conditions for the formation of a Bose-Einstein condensate (BEC) may appear in very diverse physical areas such as astronomy [15], cosmology [16], relativistic heavy-ion collisions where gluons are generated abundantly in the initial phase [17], and, in particular, cold atomic [18,19] as well as two-dimensional photonic [20] samples where BECs have actually been created experimentally.

In astronomy and cosmology, the observation of a condensate is likely beyond reach. In relativistic heavy-ion collisions such as those investigated at the Relativistic Heavy Ion Collider and the Large Hadron Collider, the creation of a gluon condensate is impeded by inelastic collisions and nonconservation of particle number [21,22], although numerical solutions of the Boltzmann equation offered some hints towards gluon condensation [23]. Cold bosonic atoms such as ^{23}Na or ^{87}Rb are a more promising tool to study nonequilibrium aspects of BEC formation. In case of cold atoms, particle-number conservation is strictly fulfilled during condensate formation.

The condensed state is reached as described in the theory of second-order phase transitions by successively decreasing the temperature and increasing the density. Although the thermodynamics of Bose condensation is thus understood quite well, it can only be applied to systems in thermal equilibrium [24–26]. Experimentally, a dilute vapor of bosonic atoms is prepared below the critical temperature T_{crit} using both laser cooling [27,28], and later magnetic evaporative cooling [18,29] to remove high-momentum atoms. Evaporation causes a compression in phase space which eventually leads to condensation of the weakly interacting cold atoms.

However, when the conditions for the formation of a condensate are reached, the bosonic system can still be far from equilibrium [30,31], and it is of interest to investigate numerical and analytical kinetic models for the time-dependent approach to the stationary state in a finite system of bosons that includes a thermal and a condensed fraction.

It is the purpose of this work to present and solve analytically such a kinetic model, and to apply it to the equilibration of an ultracold quantum gas. The statistical bosonic mode population probability $n_{\rm th}(\epsilon,t)$ is treated explicitly through the exact solution of a nonlinear equation, whereas the condensed population $n_{\rm c}(t)$ is considered indirectly through particle-number conservation, $N_{\rm tot} = N_{\rm th}(t) + N_{\rm c}(t) = \text{const.}$

The method is similar to the one used earlier for the equilibration of a finite fermion system [32]. However, for fermions, the physics of thermalization is very different since Pauli's exclusion principle determines the structure of the final state and no condensate appears. For bosons, a more detailed account of the model is given in ref. [33], with the full expression of the analytical solution, and a high-energy application to the local thermalization of the dense gluon system created initially in relativistic heavy-ion collisions.

Derivation of the transport equation. – The Boltzmann equation with the corresponding statistical factors in the collision term provides a starting point to model the time evolution of the occupation-number distribution in a finite system of bosons. The ensuing nonlinear kinetic equation preserves the essential features of Bose-Einstein statistics which are contained in the Boltzmann equation. It is then solved exactly, and applied to cold quantum gases.

Assuming spatial homogeneity for the boson distribution function $f(\boldsymbol{x},\boldsymbol{p},t)$ with momentum $\boldsymbol{p}$ or energy $\epsilon\,(\boldsymbol{p})$ and a spherically symmetric momentum dependence, one can reduce the kinetic equations to one dimension by carrying out the angular integration [34]. The equation for the single-particle occupation numbers $n_j \equiv n_{\rm th}(\epsilon_j,t)$ in a Bose system becomes

$$\frac{\partial n_1}{\partial t} = \sum_{\epsilon_2,\epsilon_3,\epsilon_4}^{\infty} \langle V_{1234}^2\rangle\, G\,(\epsilon_1+\epsilon_2,\epsilon_3+\epsilon_4) \times \big[(1+n_1)(1+n_2)\,n_3\,n_4 - (1+n_3)(1+n_4)\,n_1\,n_2\big] \quad (1)$$

with the second moment of the interaction $\langle V^2\rangle$ and the energy-conserving function G. The collision term can also be written in the form of a master equation with gain and loss terms, respectively,

$$\frac{\partial n_1}{\partial t} = (1+n_1)\sum_{\epsilon_4} W_{4\to1}\,n_4 - n_1\sum_{\epsilon_4} W_{1\to4}(1+n_4) \quad (2)$$

with the transition probability

$$W_{4\to1} = \sum_{\epsilon_2,\epsilon_3} \langle V_{1234}^2\rangle\, G\,(\epsilon_1+\epsilon_2,\epsilon_3+\epsilon_4)\,(1+n_2)\,n_3 \quad (3)$$

and $W_{1\to4}$ accordingly. The summations are then replaced by integrations, introducing the densities of states $g_j \equiv g(\epsilon_j)$ and $W_{4\to1} = W_{41}g_1, W_{1\to4} = W_{14}g_4$. Because bosons are interchangeable, we have $W_{41} = W_{14}$.

The function G ensures energy conservation. It is a δ-function for an infinite system as in the usual Boltzmann collision term where the single-particle energies are time independent,

$$G\,(\epsilon_1+\epsilon_2,\epsilon_3+\epsilon_4) \to \pi\,\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4). \quad (4)$$

In a finite system the energy-conserving function may acquire a width such that off-shell scatterings between single-particle states lying apart in energy space are possible Then $W_{14} = W_{41} = W\big[\frac{1}{2}(\epsilon_4+\epsilon_1),|\epsilon_4-\epsilon_1|\big]$ depends on the absolute value of $x = \epsilon_4 - \epsilon_1$ and is peaked at $x = 0$. It is nonlinear because of the dependence on the statistical factors in eq. (3), but here I shall explicitly treat only the nonlinearity in eq. (2) with the option to improve this by an iteration scheme.

An approximation to eq. (2) can then be obtained through a Taylor expansion of n_4 and $g_4\,n_4$ around $\epsilon_4 = \epsilon_1$ to second order. By introducing transport coefficients via moments of the transition probability

$$D = \frac{1}{2}\,g_1\int_0^\infty W(\epsilon_1,x)\,x^2\mathrm{d}x, \quad v = g_1^{-1}\frac{\mathrm{d}}{\mathrm{d}\epsilon_1}(g_1 D) \quad (5)$$

one arrives at a nonlinear partial differential equation for $n \equiv n_{\rm th}(\epsilon,t) \equiv n_{\rm th}(\epsilon_1,t)$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial\epsilon}\left[v\,n\,(1+n) - n^2\frac{\partial D}{\partial\epsilon}\right] + \frac{\partial^2}{\partial\epsilon^2}\big[D\,n\big]. \quad (6)$$

Dissipative effects are expressed through the drift term $v(\epsilon,t)$, diffusive effects through the diffusion term $D(\epsilon,t)$. In the limit of constant transport coefficients, the nonlinear boson diffusion equation for the occupation-number distribution $n(\epsilon,t)$ becomes

$$\frac{\partial n}{\partial t} = -v\,\frac{\partial}{\partial\epsilon}\big[n\,(1+n)\big] + D\,\frac{\partial^2 n}{\partial\epsilon^2}\,, \quad (7)$$

and this kinetic equation can be solved exactly. The usual thermal equilibrium distribution is a stationary solution

$$n_{\rm eq}(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/T}-1} \quad (8)$$

with the chemical potential $\mu < 0$ in a finite boson system. The equilibrium temperature is given in terms of the transport coefficients, $T = -D/v$ with $v < 0$ since the drift is towards the infrared region. An equilibrium distribution with $\mu = 0$ and a constant distribution also solve eq. (7). The particle content in the condensate [5,6,17] is $(2\pi)^3\,\delta(\boldsymbol{p})\times n_{\rm c}(t)$, with $n_{\rm c}(t)$ the number density of bosons in the condensed state.

The present analytical model does not treat the second-order phase transition to the condensate below a critical temperature $T_{\rm crit}$ explicitly since a Boltzmann-type approach cannot account for the buildup of coherence which is required for the phase transition to occur [5]. Instead, the model considers the kinetics of Bose condensation before and after the phase transition as in related numerical approaches [6,31].

In particular, a change of the number of the non-condensed particles in time can occur if one separates the transitions between the continuum states and the transitions from the continuum to the condensate (and back). A version of the collision integral with energy-conserving δ-function that separately treats collisions involving and not involving condensate atoms is given by [35]. The simplified nonlinear model outlined in this work with a finite width of the energy-conserving function and transport coefficients D, v conjectures that the population transfer from the thermal cloud to the condensate is describable by the nonlinear equation (7) through the transport coefficients, which account for the microscopic properties of the system.

For a fixed equilibrium temperature T as in the present approach, the nonequilibrium evolution according to eq. (7) pushes a certain fraction of bosons into the condensed state for sufficiently large times, or for temperatures below the critical value T_{crit}. Since eq. (7) can also be written in the form of a continuity equation

$$\frac{\partial n}{\partial t} + \frac{1}{g(\epsilon)}\frac{\partial j}{\partial \epsilon} = 0, \tag{9}$$

the probability current $j(\epsilon, t)$ is

$$j(\epsilon, t) = g(\epsilon)\Big[v\, n\,(1+n) - D\,\frac{\partial n}{\partial \epsilon}\Big]. \tag{10}$$

At $\epsilon = 0$ this corresponds to the local flow of occupation probability from the thermal cloud into the condensate if the sign of the current is negative, and from the condensate into the thermal cloud if the sign is positive. The stationary state $n_{\mathrm{stat}}(\epsilon) = n(\epsilon, t = \tau_{\mathrm{stat}})$ —that replaces the thermal equilibrium solution $n_{\mathrm{eq}}(\epsilon)$ without condensate formation— is reached for $t = \tau_{\mathrm{stat}}$, which can be computed from the condition

$$v\, n(0,\tau_{\mathrm{stat}})\,[1 + n(0,\tau_{\mathrm{stat}})] = D\,\frac{\partial n(0,\tau_{\mathrm{stat}})}{\partial \epsilon}. \tag{11}$$

Based on eq. (7), the equilibration process is driven by elastic collisions that conserve the total particle number and hence the integral over the initial distribution $n_{\mathrm{i}}(\epsilon)$ is required to agree with the integral over the asymptotic distribution that includes a condensed fraction, $n_{\mathrm{tot}}(\epsilon, t) = n_{\mathrm{th}}(\epsilon, t) + n_{\mathrm{c}}(t)$. Due to condensation, the total particle number N_{tot} at $\epsilon \to 0$ has not only a thermal fraction $N_{\mathrm{th}}(t)$, but also a condensed fraction $N_{\mathrm{c}}(t)$.

The distribution of the thermal fraction has been found to be isotropic in experiments with cold atoms [18]. In contrast, the condensate atoms, which are all described by the same macroscopic wave function, reflect anisotropies of the confining potential. The present model considers explicitly the equilibration in the thermal cloud, but not the developing coherence in the condensate, and therefore an isotropic one-dimensional approach in momentum space appears justified.

Analytical solution of the nonlinear equation. – Whereas nonlinear partial differential equations are rarely solvable in closed form, in the case of eq. (7) an analytical solution can be obtained using a method that was proposed in [32] for a finite fermion system. Although the approach is similar for bosons, the different quantum-statistical properties require a new investigation in particular regarding the transition to a Bose-Einstein condensate through the different statistical properties of the boson as compared to the fermion system. The transformation

$$n(\epsilon, t) = -\frac{D}{vP(\epsilon, t)}\frac{\partial P(\epsilon, t)}{\partial \epsilon} \tag{12}$$

reduces the nonlinear boson equation (7) to a linear diffusion equation for $P(\epsilon, t)$,

$$P_t = -vP_\epsilon + DP_{\epsilon\epsilon}, \tag{13}$$

where $P_t = \partial P/\partial t, P_\epsilon = \partial P/\partial \epsilon$. An equivalent solution of eq. (7) is possible through the linear transformation

$$n(\epsilon, t) = \frac{1}{2v}w(\epsilon, t) - \frac{1}{2} \tag{14}$$

which yields Burgers' equation [36]

$$w_t + ww_\epsilon = Dw_{\epsilon\epsilon}. \tag{15}$$

This equation has the structure of a one-dimensional Navier-Stokes equation without pressure term. It has been used to describe fluid flow and, in particular, shock waves in a viscous fluid. It can be solved through Hopf's transformation [37]

$$w(\epsilon, t) = -2D\phi_\epsilon/\phi \tag{16}$$

which reduces eq. (15) to the heat equation $\phi_t = D\phi_{\epsilon\epsilon}$.

The resulting solution of eq. (7) can then be written as

$$n(\epsilon, t) = \frac{\int_{-\infty}^{+\infty}\left(\frac{\epsilon - x}{2vt} - \frac{1}{2}\right)F(\epsilon - x, t)\,G(x)\,\mathrm{d}x}{\int_{-\infty}^{+\infty}F(\epsilon - x, t)\,G(x)\,\mathrm{d}x} \tag{17}$$

with a Gaussian part arising through the linear diffusion (or heat) equation

$$F(\epsilon - x, t) = \exp\left[-\frac{(\epsilon - x)^2}{4Dt}\right] \tag{18}$$

and an exponential function

$$G(x) = \exp\left[-\frac{1}{2D}\Big(vx + 2v\int_0^x n_{\mathrm{i}}(y)\,\mathrm{d}y\Big)\right] \tag{19}$$

that contains an integral over the initial distribution $n_{\mathrm{i}}(\epsilon)$.

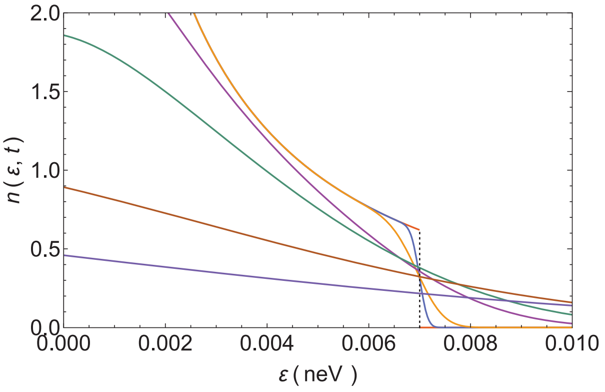

Fig. 1: (Color online) Equilibration of a finite Bose system based on the nonlinear evolution according to eq. (7) starting from a truncated equilibrium distribution as in evaporative cooling, eq. (20), upper curve with cutoff at $\epsilon_{\mathrm{i}} = 7\,\mathrm{peV}$. The transport coefficients are $D = 8 \times 10^{-3}\,\mathrm{neV}^2\,\mathrm{s}^{-1}$, $v = -1\,\mathrm{neVs}^{-1}$. The temperature $T = -D/v = 8 \times 10^{-3}\,\mathrm{neV} \simeq 93\,\mathrm{nK}$ is below the critical value for ^{87}Rb. The time sequence is 0.001, 0.01, 0.2, 0.5, 2 and 5 ms (top to bottom) with the equilibration time $\tau_{\mathrm{eq}} = 4D/(9v^2) \simeq 3.6\,\mathrm{ms}$. The nonequilibrium occupation drops below the thermal equilibrium values because the particles are redistributed into the BEC ground state in the IR, and into a new UV thermal tail.

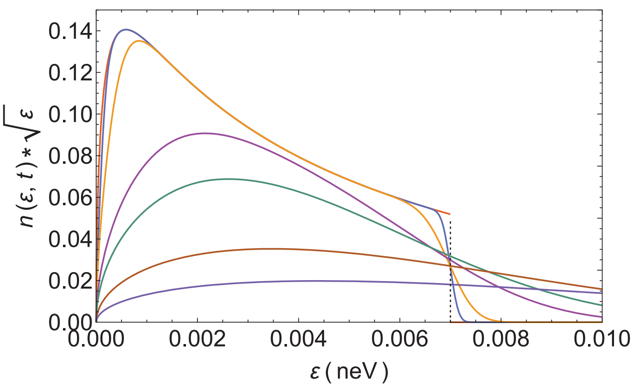

Fig. 2: (Color online) Integrands $n(\epsilon,t)\sqrt{\epsilon}$ for the nonlinear evolution according to eq. (7) with initial condition given by eq. (20), upper curve with cutoff at $\epsilon_{\mathrm{i}} = 7\,\mathrm{peV}$. The integrands are shown at six values of time t from 0.001 ms to 5 ms, top to bottom, as in fig. 1. The integrated particle number, eq. (21), in the nonequilibrium thermal cloud is not conserved during the time evolution since particles in the IR move into the $\epsilon = 0$ condensate. With increasing time a new nonequilibrium thermal tail develops in the UV.

Application to ultracold quantum gases. – The solution given by eq. (17) can be evaluated for any given nonequilibrium initial distribution. For example, when describing the equilibration in the course of evaporative cooling including BEC formation, one can start with a truncated thermal equilibrium distribution that is cut off at a maximum energy ϵ_{i}, the energy beyond which atoms have been removed as shown in fig. 1, upper curve with cutoff:

$$n_{\mathrm{i}}^{\mathrm{BEC}}(\epsilon) = N_{\mathrm{i}}\,\frac{1}{e^{(\epsilon-\mu)/T}-1}\,\theta(1-\epsilon/\epsilon_{\mathrm{i}})\,\theta(\epsilon)\,. \tag{20}$$

Numerical integration of the full kinetic equation in [29] has indeed shown that during evaporation the cold gas is characterized by a thermal distribution of atom energies, truncated at the trap depth. In the case of the analytical model solution, eq. (17), the integration is over the full energy domain $-\infty \le \epsilon \le \infty$. For bosons, $n_{\mathrm{i}}(\epsilon < 0) = 0$, whereas for fermions [32], all negative-energy states in the Dirac sea are occupied.

The time-dependent solutions, eq. (17), starting from an initial truncated thermal equilibrium distribution, eq. (20), with a cutoff energy $\epsilon_{\mathrm{i}} = 7\,\mathrm{peV}$ and $N_{\mathrm{i}} = 1$ are displayed in fig. 1 for a sequence of six times from 0.001 ms to 5 ms. The equilibrium temperature is $T = -D/v = 8 \times 10^{-3}\,\mathrm{neV} \simeq 93\,\mathrm{nK}$, below the critical value for ^{87}Rb. The values of the transport coefficients D, v are given in the caption.

The nonlinear evolution is seen to remove the discontinuities that occur in linear relaxation models at the cutoff energy $\epsilon = \epsilon_{\mathrm{i}}$. The solutions of eq. (7) display a simultaneous fall of occupation in the IR below the thermal equilibrium value $n_{\mathrm{eq}}(0) = 1/(z-1)$, with $z = \exp(-\mu/T)$ and $n_{\mathrm{eq}}(0) \simeq 11.2$ for the example shown in fig. 1 where $z = 1.089$. This fall is more pronounced at later times, and it is accompanied by a decreasing particle number in the nonequilibrium thermal cloud. The depletion occurs because the particles move into the condensed state in the IR. They are also redistributed into the emerging new UV thermal tail.

Hence, for conserved total particle number $N_{\mathrm{tot}} = N_{\mathrm{th}}(t) + N_{\mathrm{c}}(t)$ missing particles reappear as occupation of the condensate $N_{\mathrm{c}}(t) = (2\pi)^3\,\delta(\boldsymbol{p}) \times n_{\mathrm{c}}(t)$. This is expected for a Bose system below the critical temperature. Correspondingly, the particle number in the nonequilibrium thermal cloud

$$N_{\mathrm{th}}(t) = \int_0^\infty n\,(\epsilon,t)\,g\,(\epsilon)\,\mathrm{d}\epsilon \tag{21}$$

with the density of states for a cold quantum gas moving freely in three-dimensional space (as is the case for the thermal cloud [18] in a trap)

$$g\,(\epsilon) = (2m)^{3/2} V\,\sqrt{\epsilon}\,/(4\pi^2) \tag{22}$$

is diminishing with time. Here, m is the atomic mass and V the volume. This is confirmed by a numerical integration of $n\,(\epsilon,t)\,\sqrt{\epsilon}$ as displayed in fig. 2 for the same time sequence as in fig. 1.

With eq. (22) the ground state has zero weight, even though it is macroscopically occupied in the condensed phase. As is the case in equilibrium-statistical models for BEC, the condensed ground state must therefore be treated separately, which is beyond the scope of eq. (7) with initial conditions given as eq. (20): The nonlinear model describes only the time-dependent physics of the thermal cloud, including the disappearance of particles from the cloud because they move into the condensate, but not the physics of the condensate itself. This is, however, also the case for the full Boltzmann equation if no seed condensate is assumed.

An alternative solution of eq. (7) could be obtained by specifying the boundary conditions $n_b(\epsilon = 0, t) = 0$, thus forcing the system to attain zero thermal occupation at $\epsilon = 0$. Technically, in this case the integrals $\int_{-\infty}^{+\infty}$ in eq. (17) are replaced by $\int_0^{+\infty}$, and the corresponding Green's function for solving the boundary value problem at $\epsilon = 0$ replaces eq. (18). This solution would entail particle-number conservation, no particles would move into the condensed state —which is not what is observed. Hence, the solution of eq. (7) without boundary conditions at $\epsilon = 0$ is more appropriate to represent the physics in the presence of Bose-Einstein condensation.

The accuracy of the method has been tested with θ-function initial conditions

$$n_{\mathrm{i}}(\epsilon) = N_{\mathrm{i}}\, \theta(1 - \epsilon/\epsilon_{\mathrm{i}})\, \theta(\epsilon), \tag{23}$$

where the occupation is constant up to ϵ_{i}. In this case eq. (17) can be evaluated exactly [33].

A similar initial condition had been proposed at a much higher-energy scale for massless gluons with the dispersion relation $\epsilon = |\boldsymbol{p}|$ in relativistic heavy-ion collisions [38,39]. There it accounts for the early stages of a collision assuming that all gluons up to a limiting momentum are freed on a short time scale whereas all gluons beyond this saturation momentum are not freed.

The typical momentum scale is of the order of $1\,\mathrm{GeV}/c$ [40], about 20 orders of magnitude higher than the momentum scale in cold quantum gases. Yet the same formalism can in principle be applied in both cases to model the equilibration [33], with the caveat that in relativistic heavy-ion collisions, the system is rapidly expanding preferentially in the longitudinal direction. In a three-dimensional setting where isotropy is violated, the one-dimensional model, eq. (7), may be of limited usefulness. Analytically solvable extensions to two and three dimensions are, however, not conceivable, and numerical models are less transparent.

The exact evaluation [33] of eq. (17) with initial condition given by eq. (23) and the corresponding numerical result agree with high accuray, but the analytical approach is more suitable in view of further conclusions such as the derivation of an explicit expression for the equilibration time. The solution can be written as a product of exponentials and error functions, it has the expected behaviour in both the IR and UV domains, and it fulfills $n(\epsilon, t) \to n_{\mathrm{i}}(\epsilon)$ for $t \to 0$ [33].

An explicit expression for the bosonic equilibration time $\tau_{\mathrm{eq}}^{\mathrm{Bose}}$ follows from an asymptotic expansion of the error functions

$$\operatorname{erf}(z_{\mathrm{b}}) \simeq 1 - \frac{1}{\sqrt{\pi}\, z_{\mathrm{b}}} \exp\bigl[-z_{\mathrm{b}}^2\bigr] + \exp(-z_{\mathrm{b}}^2)\, \mathcal{O}\Bigl(\frac{1}{z_{\mathrm{b}}^3}\Bigr) \tag{24}$$

which occur in the analytical solution [33] at the boundary $x_{\mathrm{b}} = \epsilon_{\mathrm{i}}$,

$$z_{\mathrm{b}} = \frac{1}{2\sqrt{Dt}} \bigl[x_{\mathrm{b}} - \epsilon + (1 + 2N_{\mathrm{i}})\, vt\bigr]. \tag{25}$$

Therefore, deviations from the asymptotic solution in the thermal tail of the distribution function scale with $\exp[-(1+2N_{\mathrm{i}})^2 v^2 t/(4D)]$ such that the equilibration time in a system of bosons with an initial distribution given by eq. (23) becomes

$$\tau_{\mathrm{eq}}^{\mathrm{Bose}} = 4D/[(1 + 2N_{\mathrm{i}})^2\, v^2]. \tag{26}$$

As a consequence, the Bose equilibration time for $N_{\mathrm{i}} = 1$ is a factor nine shorter than the corresponding equilibration time in a fermion system, which was found to be $\tau_{\mathrm{eq}}^{\mathrm{Fermi}} = 4D/v^2$ in [32]. This difference is solely due to the quantum-statistical properties of a boson as compared to a fermion system: In a fermion system, changes of the occupation of single-particle states are suppressed due to the exclusion principle and consequently, the equilibration process takes more time for fermions than for bosons.

Hence, short equilibration times encountered in bosonic systems —such as the initial state in relativistic heavy-ion collisions that is determined mostly by gluons— are to a large extent due to the statistical factors for bosons, as the appearance of a phase transition in bosonic systems is due to the particle correlations imposed by Bose statistics. It is interesting that $\tau_{\mathrm{eq}}^{\mathrm{Bose}} \sim 1/(1 + 2N_{\mathrm{i}})^2$ decreases with rising occupation N_{i} of the initial state. In confined geometries the equilibration time may be somewhat different from three-dimensional systems where the thermal cloud is isotropic in momentum space, because spatially confined geometries induce anisotropies in the momentum distribution of the thermal cloud such that there is not a single local equilibration time characterizing the system.

There is presently no microscopic calculation of the transport coefficients available. However, since the model uniquely connects v and D with T and τ_{eq}, fixed values of the latter have been chosen to compute v, D.

Obviously, such a model with constant transport coefficients is an idealization that is motivated by the possibility to find an exact solution of the dynamics of the thermal cloud as described by a kinetic Boltzmann equation. From the microscopic structure as given in eq. (5), the diffusion coefficient D is constant if $g_1 W_{41}$ and, therefore, the rate $W_{4\to 1}$ of eq. (3) in the master equation (2) is independent of energy, which for contact interaction is true only in the high-temperature limit when the occupations n_2 and n_3 are independent of energy[1]. For constant D, any energy dependence of the drift according to eq. (5) is due to the single-particle level density g_1, and constant v would require an exponential energy dependence of g. Hence, the transport coefficient functions v, D certainly call for further detailed investigations starting from the microscopic

[1]This was observed by one of the referees.

structure of the transition probabilities, which is, however, beyond the scope of this letter which relies on constant coefficients to provide an exact solution.

Conclusion. – To summarize, I have outlined a schematic model for equilibration in finite Bose systems. The master equation for the bosonic mode occupation probabilities has been transformed into a nonlinear partial differential equation that keeps track of the statistical factors in an essential way: It allows the system to evolve into the condensate in the infrared region, and to develop a thermal tail in the ultraviolet. A closed-form solution has been obtained in the simplified case of constant transport coefficients. I have applied the model to the equilibration of a cold quantum gas such as ^{87}Rb below the critical temperature, with a truncated thermal equilibrium distribution as initial condition that corresponds to the situation encountered during evaporative cooling.

The phase transition is not accounted for explicitly in the model, but indirectly through particle-number conservation. Hence, the model describes only the time-dependent physics of the thermal cloud, including the disappearance of particles from the cloud because they move into the condensate —not the physics of the condensate itself.

Dissipative and diffusive effects combine with the nonlinearity to yield the time evolution towards the asymptotic thermal distribution $n_{\text{stat}}(\epsilon)$, which differs from the usual thermal equilibrium limit $n_{\text{eq}}(\epsilon)$ since part of the system reaches the condensed state at $\epsilon = 0$. Simultaneously, a thermal tail develops within the bosonic equilibration time. This is different from fermionic systems where all particles attain the thermal equilibrium limit at large times [32].

* * *

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